COMBINATORIAL REES–SUSHKEVICH VARIETIES ARE FINITELY BASED

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A variety is said to be a Rees–Sushkevich variety if it is contained in a periodic variety generated by 0-simple semigroups. It is shown that all combinatorial Rees–Sushkevich varieties are finitely based.

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1. Introduction

1.1. Rees–Sushkevich varieties

The collection $S$ of all semigroup varieties forms a complete lattice with respect to inclusion. It is known that the lattice $S$ has a very complex structure. To a certain extent, the complexity of $S$ is maximal possible since it follows from [2] that the subvariety lattice of any variety of algebras of finite type embeds into $S$. Consequently, the study of $S$ constitutes a highly nontrivial problem. However, some regions in $S$ are relatively well-understood by now.

A semigroup variety that contains the variety of all commutative semigroups is said to be overcommutative. The lattice $S$ consists of the sublattice $\mathbb{C}$ of overcommutative varieties and the sublattice $P$ of periodic varieties. The lattice $\mathbb{C}$ turns out to be a filter that admits a relatively simple description in terms of congruence lattices of certain unary algebras [33]. Within the lattice $P$ lies two important ideals formed by completely regular varieties and nilsemigroup varieties. These two ideals were respectively investigated by Polák [17–19] and Vernikov and Volkov [28–30].
Any subvariety of a periodic variety generated by 0-simple semigroups is said to be a Rees–Sushkevich variety [8]. It follows from the preceding discussion that any further progress in understanding the lattice \( \mathcal{S} \) requires the exploration of periodic varieties containing 0-simple semigroups with zero divisors. It is thus natural to begin this task by investigating the lattice of Rees–Sushkevich varieties. This lattice has recently been studied by Kublanovsky, Reilly, Volkov, and the author (see, for example, [8–16, 20–23, 34]).

A semigroup variety is *combinatorial* if all groups in it are trivial. The present article is concerned with the finite basis property of all combinatorial Rees–Sushkevich varieties.

**Theorem 1.1.** *Every combinatorial Rees–Sushkevich variety is finitely based.*

For any nontrivial periodic group \( G \), there exist non-finitely based 0-simple semigroups with subgroups isomorphic to \( G \) [32]. Therefore Theorem 1.1 has reached its full potential in the sense that for each integer \( n \geq 2 \), not all Rees–Sushkevich varieties with subgroups of exponent dividing \( n \) are finitely based.

1.2. **The variety \( A_2 \)**

Denote by \( A_2 \) the variety generated by the 0-simple semigroup

\[
A_2 = \langle a, b : a^2 = aba = a, b^2 = 0, bab = b \rangle
\]

of order five. Since the semigroup \( A_2 \) is 0-simple and has no nontrivial subgroups, the variety \( A_2 \) is a combinatorial Rees–Sushkevich variety. This variety, despite being generated by a semigroup with very few elements, is crucial to the present investigation.

**Proposition 1.2.** *The variety \( A_2 \) is the largest combinatorial Rees–Sushkevich variety.*

**Proof.** It is well-known that any periodic 0-simple semigroup is completely 0-simple, and that such a semigroup is isomorphic to some Rees matrix semigroup (see, for example, [6, Sec. 3.2]). It follows from [5, proof of Theorem 3.3] that any combinatorial Rees matrix semigroup belongs to the variety \( A_2 \).

Recall that a variety is *hereditarily finitely based* if all its subvarieties are finitely based. It is then easy to see that Theorem 1.1 is equivalent to the following result.

**Theorem 1.3.** *The variety \( A_2 \) is hereditarily finitely based.*

The proof of Theorem 1.3 is divided into several parts and presented in Secs. 3 and 4. Further details can be found in Remark 2.8.

An immediate consequence of a variety being hereditarily finitely based is that it cannot contain “too many” subvarieties. Indeed, since only countably many finite sets of identities exist up to letter substitution, a hereditarily finitely based variety
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contains at most countably many subvarieties. The following is thus a consequence of Theorem 1.3 the first part of which answers a question of Jackson [7].

**Corollary 1.4.** The variety $A_2$ contains only countably many subvarieties. Equivalently, the lattice of combinatorial Rees–Sushkevich varieties is countably infinite.

Despite having a bound on the number of subvarieties of $A_2$, the structure of the lattice $\mathcal{L}(A_2)$ of subvarieties of $A_2$ is quite complex since it contains an isomorphic copy of every finite lattice [29].

In the presence of the hereditary properties of the variety $A_2$ in Theorem 1.3 and Corollary 1.4, it is somewhat interesting to note that by merely adjoining an identity element to its generating semigroup $A_2$, the resulting monoid $A_1^2$ generates a variety with very contrasting properties. More specifically, the variety generated by $A_1^2$ is inherently non-finitely based [24] and contains uncountably many subvarieties [26].

2. Background

2.1. Notation

Let $X^+$ and $X^*$ respectively be the free semigroup and free monoid over a countably infinite alphabet $X$. Elements of $X$ are referred to as letters and elements of $X^+$ and $X^*$ are referred to as words.

Let $w$ be any word. The head and tail of $w$ are the first and last letters occurring in $w$ and are denoted by $h(w)$ and $t(w)$ respectively. The length of $w$, denoted by $|w|$, is the number of letters occurring in $w$ counting multiplicity. The content of $w$ is the set of letters occurring in $w$ and is denoted by $C(w)$. The words $w_1, \ldots, w_m$ are said to be disjoint if the sets $C(w_1), \ldots, C(w_m)$ are pairwise disjoint. The set of factors of $w$ of length two is denoted by $C_2(w)$.

An identity is typically written as $u \approx v$ where $u, v \in X^+$. Let $\Sigma$ be any set of identities. The deducibility of an identity $u \approx v$ from $\Sigma$ is indicated by $\Sigma \vdash u \approx v$ or $u \Sigma \approx v$. The variety defined by $\Sigma$ is the class of all semigroups that satisfy all identities in $\Sigma$; in this case, $\Sigma$ is said to be a basis for the variety. If $V$ is a variety, then the subvariety of $V$ defined by $\Sigma$ is denoted by $V\{\Sigma\}$. The reader is referred to [3] for other undefined notation and terminology in this article.

The number $|C(w)|$ of distinct letters in a word $w$ obviously cannot exceed the length $|w|$ of $w$, whence the difference $|w| - |C(w)|$ is always a nonnegative integer; define this integer to be the level of $w$.

**Lemma 2.1 (Volkov [31]).** Let $m$ be a fixed integer. Suppose that $\Sigma$ is any set of identities each of which is formed by a pair of words of level at most $m$. Then the variety defined by $\Sigma$ is finitely based.

**Lemma 2.2 (Trahtman [25,27]).** The identities

$$x^3 \approx x^2, \ xyxyx \approx xyx, \ xyxxz \approx zyxy$$  \hspace{1cm} (1)
constitute a basis for the variety $A_2$. More generally, an identity $u \approx v$ holds in the variety $A_2$ if and only if $C_2(u) = C_2(v)$, $h(u) = h(v)$, and $t(u) = t(v)$.

In the presence of Lemma 2.2, for any two words $u$ and $v$, the deduction $u \vdash v$ holds if and only if $C_2(u) = C_2(v)$, $h(u) = h(v)$, and $t(u) = t(v)$. Throughout this article, identities of $A_2$ will often be recognized in this manner without further reference to the lemma.

Lemma 2.3. If $h(u) = h(v)$ and $t(u) = t(v)$, then $A_2 \models uv \approx vu$.

Proof. This follows from Lemma 2.2 since $C_2(uv) = C_2(vu)$.

2.2. Some subvarieties of $A_2$

The semigroup $A_2$ and the combinatorial Brandt semigroup

$$B_2 = \langle c, d : c^2 = d^2 = 0, cde = c, cdc = d \rangle$$

of order five are, up to isomorphism, the only minimal examples of 0-simple semigroups with zero divisors. It is straightforward to verify that the sets

$$A_0 = \{0, b, ab, ba\} \text{ and } B_0 = \{0, d, db, dc\}$$

are subsemigroups of $A_2$ and $B_2$ respectively. Denote by $A_0$, $B_0$, and $B_2$ the varieties generated by the semigroups $A_0$, $B_0$, and $B_2$ respectively. It follows from Proposition 1.2 that $A_0$, $B_0$, and $B_2$ are subvarieties of $A_2$. In the lattice of semigroup varieties, the position of these three subvarieties of $A_2$ are very close.

Lemma 2.4 (Lee [10]). The variety $B_0$ is the unique maximal proper subvariety of both $A_0$ and $B_2$. In particular, $A_0 \cap B_2 = B_0$.

For any subvariety $V$ of $A_2$, the subvariety of $A_2$ which is largest with respect to not containing $V$, if it exists, is denote by $\overline{V}$.

Proposition 2.5. The varieties $\overline{A_2}$, $\overline{B_2}$, $\overline{A_0}$, and $\overline{B_0}$ exist and are all finitely based. More specifically,

(i) $\overline{A_2} = A_2\{x^2y^2x^2 \models x^2y^2x^2\}$;
(ii) $\overline{B_2} = A_2\{xy^2x \models xyx\}$;
(iii) $\overline{A_0} = A_2\{x^2y^2x^2 \models x^2y^2\}$;
(iv) $\overline{B_0} = A_2\{x^2y^2z^2 \models x^2y^2z^2\}$.

Proof. Parts (i) and (ii) are from [10, Theorems 2.7 and 3.6]. Part (iii) follows from [9, Lemma 2.7]. Part (iv) will be proved in Subsec. 3.2.

Lemma 2.6. The inclusions $\overline{B_0} \subseteq \overline{A_0} \subseteq \overline{A_2}$ and $\overline{B_0} \subseteq \overline{B_2} \subseteq \overline{A_2}$ hold.

Proof. If $U \subseteq V \subseteq A_2$ and $\overline{U}$ and $\overline{V}$ exist, then it is easy to show that $\overline{U} \subseteq \overline{V}$. The result now follows from Lemma 2.4 and Proposition 2.5.
Proposition 2.7. The lattice $\mathcal{L}(\overline{A}_2)$ is the disjoint union of the following intervals:

$$
\mathcal{I}_1 = [A_0 \vee B_2, \overline{A}_2],
\mathcal{I}_2 = [A_0, \overline{B}_2],
\mathcal{I}_3 = [B_2, \overline{A}_0],
\mathcal{I}_4 = [B_0, \overline{A}_0 \cap \overline{B}_2],
\mathcal{I}_5 = \mathcal{L}(\overline{B}_0).
$$

Proof. It is easy to show that the intervals $\mathcal{I}_1, \ldots, \mathcal{I}_5$ are pairwise disjoint. Let $V \in \mathcal{L}(\overline{A}_2)$. If $A_0, B_2 \in V$, then $V \in \mathcal{I}_1$. If precisely one semigroup from $\{A_0, B_2\}$ belongs to $V$, then either $V \in \mathcal{I}_2$ or $V \in \mathcal{I}_3$. Therefore assume that $A_0, B_2 \notin V$, whence $V \in \mathcal{L}(\overline{A}_0 \cap \overline{B}_2)$. Now $V$ belongs to $\mathcal{I}_4$ or $\mathcal{I}_5$ depends on whether or not it contains $B_0$. \qed

Remark 2.8. All varieties in the intervals $\mathcal{I}_1$ and $\mathcal{I}_2$ are shown to be finitely based in Sec. 3 (Propositions 3.5 and 3.14) while those in the intervals $\mathcal{I}_3, \mathcal{I}_4$, and $\mathcal{I}_5$ are shown to be finitely based in Sec. 4 (Propositions 4.3, 4.4, and 4.6). In the presence of Lemma 2.2 and Proposition 2.7, the proof of Theorem 1.3 is complete.

3. The Intervals $\mathcal{I}_1$ and $\mathcal{I}_2$

3.1. Identities of varieties in $\mathcal{I}_1$ and $\mathcal{I}_2$

In this subsection, restrictions on identities that can be used to define varieties in the interval $\mathcal{I}_1 \cup \mathcal{I}_2 = [A_0, \overline{A}_2]$ are given. These restrictions are used in Subsecs. 3.2 and 3.4 to show that all varieties in the intervals $\mathcal{I}_1$ and $\mathcal{I}_2$ are finitely based.

A word is simple if all letters occurring in it have multiplicity one. A word of length one is a singleton word. A word of length at least two is said to be connected if it cannot be written as a product of two disjoint nonempty words.

Lemma 3.1. Let $u = u_1 \cdots u_m$ (respectively, $v = v_1 \cdots v_n$) where $u_1, \ldots, u_m$ (respectively, $v_1, \ldots, v_n$) are disjoint words each of which is either connected or singleton.

(i) If $A_0 \models u \approx v$, then $m = n$ and $A_0 \models u_i \approx v_i$ for all $i$.
(ii) If $A_0 \vee B_2 \models u \approx v$, then $m = n$ and $A_0 \vee B_2 \models u_i \approx v_i$ for all $i$.

Proof. This follows from [15, Proposition 3.2] and its dual result. \qed

Consider a variety in $\mathcal{I}_1 \cup \mathcal{I}_2 = [A_0, \overline{A}_2]$ of the form $\overline{A}_2(\xi)$ where $\xi : u \approx v$ is a nontrivial identity of $A_0$. If either $u$ or $v$ is simple, then it follows from Lemma 3.1(i) that $u$ and $v$ are contradictorily identical. Therefore both $u$ and $v$ are non-simple. It is easy to see that each non-simple word can be uniquely written as a product of disjoint words each of which is either connected or singleton.
Therefore
\[ u = s_1 u_1 s_2 u_2 \cdots s_k u_k s_{k+1} \quad \text{and} \quad v = s_1 v_1 s_2 v_2 \cdots s_k v_k s_{k+1} \]
by Lemma 3.1(i), where

(A1) \( s_1, \ldots, s_{k+1} \in X^* \) are simple;
(A2) \( u_1, \ldots, u_k, v_1, \ldots, v_k \in X^* \) are connected;
(A3) \( s_1, u_1, \ldots, s_k, u_k, s_{k+1} \) are disjoint;
(A4) \( s_1, v_1, \ldots, s_k, v_k, s_{k+1} \) are disjoint;
(A5) \( A_0 = u_i \approx v_i \) (so that \( C(u_i) = C(v_i) \)) for all \( i \).

For any \( \ell \) and \( r \) from the set
\[ \mathbb{N}^* = \{0, 1, 2, \ldots, \omega, \omega + 1\}, \]
define the words
\[
 a_{(\ell)} = \begin{cases} 
 \emptyset & \text{if } \ell = 0, \\
 a_1 \cdots a_\ell & \text{if } 1 \leq \ell < \omega, \\
 a_1^2 a_2 & \text{if } \ell = \omega, \\
 a_1^2 & \text{if } \ell = \omega + 1. 
\end{cases}
\]
and
\[
 b_{(r)} = \begin{cases} 
 \emptyset & \text{if } r = 0, \\
 b_1 \cdots b_r & \text{if } 1 \leq r < \omega, \\
 b_1 b_2^2 & \text{if } r = \omega, \\
 b_2^2 & \text{if } r = \omega + 1. 
\end{cases}
\]

In what follows, a finite set \( \Pi_\xi \) of identities is constructed with the property that
\[
 A_2(\xi) = A_2(\Pi_\xi). \tag{2}
\]
Each identity in \( \Pi_\xi \) will be of the form \( a_{(\ell)} u_i b_{(r)} \approx a_{(\ell)} v_i b_{(r)} \) where \( \ell, r \in \mathbb{N}^* \) and \( u_i \) and \( v_i \) are words from (A2) that do not contain any letters from \( C(a_{(\ell)} b_{(r)}) \).

First consider the case when \( k = 1 \). Then the identity \( \xi \) is \( s_1 u_1 s_2 \approx s_1 v_1 s_2 \) so that the equation in (2) holds with \( \Pi_\xi = \{ a_{(\ell)} u_1 b_{(r)} \approx a_{(\ell)} v_1 b_{(r)} \} \), where \( \ell = |s_1| \) and \( r = |s_2| \).

Now suppose that \( k \geq 2 \). Denote by \( \gamma_1 \) the substitution
\[ x \mapsto \begin{cases} b_1 & \text{if } x = h(s_2), \\
 b_2^2 & \text{if } x \text{ occurs after } h(s_2) \text{ in } u. 
\end{cases} \]
Then
\[
 \{(1), \xi \} \vdash s_1 u_1 b_1 b_2^2 \overset{(1)}{=} s_1 v_1 b_1 b_2^2 \
 \vdash \xi : s_1 u_1 b_1 b_2^2 \approx s_1 v_1 b_1 b_2^2
\]
where \( b_1 = \emptyset \) if and only if \( s_2 = \emptyset \), whence the identity \( \xi_1 \) is \( a_{(\ell)} u_1 b_{(r)} \approx a_{(\ell)} v_1 b_{(r)} \) with \( \ell = |s_1| \) and \( r \in \{\omega, \omega + 1\} \). For \( 1 < i < k \), denote by \( \gamma_i \) the substitution
\[ x \mapsto \begin{cases} a_1^2 & \text{if } x \text{ occurs before } t(s_i) \text{ in } u, \\
 a_2 & \text{if } x = t(s_i), \\
 b_1 & \text{if } x = h(s_{i+1}), \\
 b_2^2 & \text{if } x \text{ occurs after } h(s_{i+1}) \text{ in } u. 
\end{cases} \]
Then

\[
\{(1), \xi \} \vdash a_1^2a_2u_kb_1b_2^2 \overset{(1)}{=} u_\gamma_i \overset{(1)}{=} a_1^2a_2v_i b_1b_2^2
\]

where \( a_3 = \emptyset \) if and only if \( s_i = \emptyset \), and \( b_1 = \emptyset \) if and only if \( s_{i+1} = \emptyset \), whence the identity \( \xi_i \) is \( a_1(\xi)u_i b_1(\xi) \approx a_1(\xi)v_i b_1(\xi) \) with \( \ell, r \in \{\omega, \omega + 1\} \). Finally, denote by \( \gamma_k \) the substitution

\[
x \mapsto \begin{cases} 
  a_1^2 & \text{if } x \text{ occurs before } t(s_k) \text{ in } u, \\
  a_2 & \text{if } x = t(s_k).
\end{cases}
\]

Then

\[
\{(1), \xi \} \vdash a_1^2a_2u_k s_{k+1} \overset{(1)}{=} u_\gamma_k \overset{(1)}{=} a_1^2a_2v_k s_{k+1}
\]

where \( a_2 = \emptyset \) if and only if \( s_k = \emptyset \), whence the identity \( \xi_k \) is \( a_1(\xi)u_k b_1(\xi) \approx a_1(\xi)v_k b_1(\xi) \) with \( \ell \in \{\omega, \omega + 1\} \) and \( r = |s_{k+1}| \). Therefore the deduction \( \{(1), \xi \} \vdash \{\xi_1, \ldots, \xi_k\} \) holds.

It remains to establish the deduction \( \{(1), \xi_1, \ldots, \xi_k\} \vdash \xi \) so that the equation in (2) holds for \( \Pi_\xi = \{\xi_1, \ldots, \xi_k\} \). For each \( i \), the word \( u_i \) is connected so that the letter \( h = h(u_i) \) occurs at least twice in \( u_i \). Therefore \( u_i = hu_i'hu_i'' \) for some \( u_i' \in \mathcal{X}^+ \), whence \( u_i \overset{(1)}{=} hu_i'^2u_i \). Thus \( u_i \overset{(1)}{=} e_f^2u_i \) for some \( e_f \in \mathcal{X}^+ \). Similarly, the letter \( t(v_i) \) occurs at least twice in \( v_i \) so that \( v_i \overset{(1)}{=} v_i f_i^2 \) for some \( f_i \in \mathcal{X}^+ \). Therefore

\[
\{(1), \xi_1, \ldots, \xi_k\} \vdash u = s_1u_1s_2u_2s_3u_3 \cdots s_ku_k s_{k+1}
\]

\[
\overset{(1)}{\approx} (s_1u_1s_2e_f^2)u_2s_3u_3 \cdots s_ku_k s_{k+1}
\]

\[
\overset{\xi_1}{\approx} s_1v_1s_2(e_f^2)u_3 \cdots s_ku_k s_{k+1}
\]

\[
\overset{(1)}{\approx} s_1v_1(s_f^2u_2s_3e_f^2)u_3 \cdots s_ku_k s_{k+1}
\]

\[
\overset{\xi_2}{\approx} s_1v_1(s_f^2)u_2s_3(e_f^2)u_3 \cdots s_ku_k s_{k+1}
\]

\[
\overset{(1)}{\approx} s_1v_1-s_{k-1}v_{k-1}(e_f^2)u_3 \cdots s_ku_k s_{k+1}
\]

\[
\overset{\xi_k}{\approx} s_1v_1-s_{k-1}v_{k-1}(e_f^2)u_3 \cdots s_ku_k s_{k+1}
\]

\[
= v
\]

as required.
Lemma 3.2. Each variety in the interval \( I_1 \) (respectively, \( I_2 \)) is defined within \( \overline{A}_2 \) (respectively, \( \overline{B}_2 \)) by identities of the form

\[
a_{(\ell)} u_{(r)} b_{(r)} \approx a_{(\ell)} v_{(r)}
\]

(3)

where

(a) \( \ell, r \in \mathbb{N}^* \); 
(b) \( u, v \in X^+ \) are connected;
(c) \( a_{(\ell)}, u, b_{(r)} \) are disjoint;
(d) \( a_{(\ell)}, v, b_{(r)} \) are disjoint;
(e) \( A_0 \vee B_2 \models u \approx v \) (respectively, \( A_0 \models u \approx v \)).

Proof. Suppose that \( V \in I_1 \). Then \( V = \overline{A}_2 \{ \Sigma \} \) for some set \( \Sigma \) of identities of \( A_0 \vee B_2 \). Let \( \xi \) be any identity in \( \Sigma \), which can be assumed to be the one following Lemma 3.1 that satisfies conditions (A1) to (A5). By Lemma 3.1(ii), the additional condition that \( B_2 \models u_i \approx v_i \) for \( 1 \leq i \leq k \) can also be assumed. Repeating the argument following the proof of Lemma 3.1 results in \( \overline{A}_2 \{ \xi \} = \overline{A}_2 \{ \Pi_\xi \} \) where the identities in \( \Pi_\xi \) are of the form (3) that satisfy conditions (a) to (e). Replace each identity \( \xi \) in \( \Sigma \) by the identities in \( \Pi_\xi \) to obtain the desired result.

The result regarding varieties in the interval \( I_2 \) follows very similarly. \( \square \)

3.2. The finite basis property for varieties in \( I_2 \)

Lemma 3.3. Let \( u \) and \( v \) be any connected words. Then \( \overline{B}_2 \models u \approx v \) if and only if \( C(u) = C(v) \), \( h(u) = h(v) \), and \( t(u) = t(v) \).

Proof. Suppose that \( C(u) = C(v) = \{ x_1, \ldots, x_k \} \), \( h(u) = h(v) \), and \( t(u) = t(v) \). Let \( F_k \) be the free object of \( \overline{B}_2 \) over the generators \( \{ x_1, \ldots, x_k \} \). Then \( u \) and \( v \) are regular elements in \( F_k \) by [15, Proposition 2.2]. Since \( A_2 \) is a locally finite variety, the semigroup \( F_k \) is finite. It follows from [1, Theorem 8.1.7] and the remark following its proof that the regular elements \( u \) and \( v \) of \( F_k \) are \( D \)-related. However, by Lemma 2.3, the elements \( u \) and \( v \) commute in \( F_k \) and so must coincide in \( F_k \).

Conversely, if \( \overline{B}_2 \models u \approx v \), then \( C(u) = C(v) \), \( h(u) = h(v) \), and \( t(u) = t(v) \) since the variety \( \overline{B}_2 \) contains semilattices, left-zero semigroups, and right-zero semigroups. \( \square \)

Lemma 3.3 is a very convenient tool for recognizing identities of \( \overline{B}_2 \). Some useful identities of \( \overline{B}_2 \) are

\[
\begin{align*}
  xhxt &\approx x^2hxt \approx xhy^2tx \approx xhytx^2, \\
  xhpy &\approx x^2hpy \approx xhy^2z \approx xhyxz^2, \\
  xytx &\approx x^2yttx \approx x^2tx \approx xytz^2, \\
  xryt &\approx x^2ryt \approx x^2tx \approx xryz^2.
\end{align*}
\]

(4)
It follows from Proposition 2.5(ii) that the identities \((1, 4)\) also constitute a basis for the variety \(\overline{B}_2\).

**Proof of Proposition 2.5(iv).** Let \(V\) be any subvariety of \(A_2\) that does not contain the semigroup \(B_0\). Let \(S\) be any finite semigroup in \(V\). The variety generated by \(S\) clearly does not contain the semigroup \(B_2\) so that \(S\) satisfies the identities \((4)\). It follows from [1, Proposition 11.8.1] that \(S\) also satisfies one of the identities
\[
\epsilon_1 : x^2yz^2 \approx (x^2yz)^2,
\epsilon_2 : x^2yz^2 \approx x^2yz(x^2z^2)^2,
\epsilon_3 : x^2yz^2 \approx (x^2z^2)^2x^2yz^2.
\]
For \(i \in \{1, 2, 3\}\), denote by \(w_i\) the word on the right side of the identity \(\epsilon_i\). The letter \(y\) of \(w_i\) is sandwiched between two occurrences of either \(x\) or \(z\) so that the deduction \((4) \vdash w_i \varphi \approx w_i\) holds under the substitution \(y \mapsto y^2\). Therefore
\[
\{\epsilon_i, (4)\} \vdash x^2yz^2 \approx w_i \overset{(4)}{=} w_i \varphi \overset{\varphi}{=} (x^2yz)^2 \varphi = x^2y^2z^2 \vdash \epsilon : x^2y^2z^2 \approx x^2yz^2,
\]
whence the identity \(\epsilon\) holds in the semigroup \(S\). Since \(S\) is an arbitrary finite semigroup in the locally finite variety \(V\), the identity \(\epsilon\) also holds in \(V\). \(\square\)

A simple word \(x_1 \cdots x_m\) is said to be an **ordered word** if the letters \(x_1, \ldots, x_m\) are in alphabetical order. A word \(w\) is said to be a \((4)\)-**word** if either

\[(B1)\] \(w = xu \bar{x}\) for some ordered word \(u \in X^*\) with \(x \notin C(u)\), or
\[\text{(B2)}\] \(w = xy \bar{y} xu\) for some ordered word \(u \in X^*\) with \(x, y \notin C(u)\) and \(x \neq y\).

Note that any \((4)\)-word is connected and of level at most two.

**Lemma 3.4.** Let \(w\) be any connected word. Then the deduction \((1, 4) \vdash w \approx w'\) holds for some unique \((4)\)-word \(w'\) such that \(C(w) = C(w')\), \(h(w) = h(w')\), and \(t(w) = t(w')\).

**Proof.** This is an easy consequence of Lemma 3.3. \(\square\)

**Proposition 3.5.** Each variety in the interval \(I_2 = [A_0, \overline{B}_2]\) is finitely based.

**Proof.** Suppose that \(V \in I_2\). Then it follows from Lemma 3.2 that \(V = \overline{B}_2\{\Sigma\}\) for some set \(\Sigma\) of identities of the form \(a_{(r)}ub_{(r)} \approx a_{(r)}vb_{(r)}\) that satisfy conditions (a) to (e) in that lemma. By Lemma 3.4 and since the identities \((1, 4)\) hold in the variety \(\overline{B}_2\), the words \(u\) and \(v\) can be chosen to be \((4)\)-words. Therefore the identities in \(\Sigma\) are formed by words of level at most four, whence the variety defined by \(\Sigma\) is finitely based by Lemma 2.1. It follows from Proposition 2.5 that the variety \(V = \overline{B}_2\{\Sigma\}\) is also finitely based. \(\square\)
3.3. **Refinement of identities of varieties in \( \mathcal{I}_2 \)**

The ability of the identities of \( \mathcal{B}_2 \) to identify each connected word with a unique (4)-word of uniformly bounded level enabled the application of Volkov's theorem (Lemma 2.1) in the proof of Proposition 3.5. However, the identities of \( \mathcal{B}_2 \) are not strong enough to perform a similar task in \( \mathcal{I}_1 \). In this subsection, the form of identities that can be used to define varieties in the interval \( \mathcal{I}_2 \) is refined (Lemma 3.8). This is then extended in the next subsection to identities of varieties in the interval \( \mathcal{I}_1 \) with the eventual aim of proving the finite basis property for all varieties in \( \mathcal{I}_1 \).

For any \( \ell, r \in \mathbb{N}^* \), define the identities

\[
\mu_{\ell,r} : a(\ell)x^2y^2x^2b(r) \approx a(\ell)y^2x^2y^2b(r), \\
\lambda_{\ell,r} : a(\ell)x^2y^2x^2b(r) \approx a(\ell)y^2x^2y^2b(r), \\
\rho_{\ell,r} : a(\ell)x^2y^2x^2b(r) \approx a(\ell)y^2x^2y^2b(r).
\]

**Lemma 3.6.** The equation \( \mathcal{A}_2\{\mu_{\ell,r}\} = \mathcal{A}_2\{\lambda_{\ell,r}, \rho_{\ell,r}\} \) holds for all \( \ell, r \in \mathbb{N}^* \).

**Proof.** The inclusion \( \mathcal{A}_2\{\lambda_{\ell,r}, \rho_{\ell,r}\} \subseteq \mathcal{A}_2\{\mu_{\ell,r}\} \) holds since

\[
\{\lambda_{\ell,r}, \rho_{\ell,r}\} \vdash a(\ell)x^2y^2x^2b(r) \overset{\lambda_{\ell,r}}{=} a(\ell)y^2x^2y^2b(r) \overset{\rho_{\ell,r}}{=} a(\ell)y^2x^2y^2b(r) \vdash \mu_{\ell,r}.
\]

Conversely, the inclusion \( \mathcal{A}_2\{\mu_{\ell,r}\} \subseteq \mathcal{A}_2\{\lambda_{\ell,r}, \rho_{\ell,r}\} \) holds since

\[
\{(1), \mu_{\ell,r}\} \vdash a(\ell)x^2y^2x^2b(r) \overset{1}{=} (a(\ell)x^2y^2x^2 \cdot x^{b(r)})x^2b(r) \overset{\mu_{\ell,r}}{=} a(\ell)y^2x^2y^2b(r) \overset{1}{=} a(\ell)y^2x^2y^2b(r) \vdash \lambda_{\ell,r}
\]

and \( \{(1), \mu_{\ell,r}\} \vdash \rho_{\ell,r} \) by symmetry. \( \square \)

**Lemma 3.7.** Let \( \sigma : u \approx v \) be any identity of \( \mathcal{A}_0 \) where \( u \) and \( v \) are distinct (4)-words. Then

\[
\mathcal{B}_2\{\sigma\} = \begin{cases} 
\mathcal{B}_2\{\mu_{0,0}\} & \text{if } h(u) \neq h(v) \text{ and } t(u) \neq t(v), \\
\mathcal{B}_2\{\lambda_{0,0}\} & \text{if } h(u) \neq h(v) \text{ and } t(u) = t(v), \\
\mathcal{B}_2\{\rho_{0,0}\} & \text{if } h(u) = h(v) \text{ and } t(u) \neq t(v).
\end{cases}
\]

**Proof.** Recall from Lemma 3.3 and the remark following it that if \( w_1 \) and \( w_2 \) are any connected words, then the deduction \( w_1 \overset{(1,4)}{=} w_2 \) holds if and only if \( C(w_1) = C(w_2), h(w_1) = h(w_2), \) and \( t(w_1) = t(w_2) \). This fact will be used very frequently in this proof.
Case 1. \( u \) and \( v \) are both of the form (B2).

1.1. \( h(u) \neq h(v) \) and \( t(u) \neq t(v) \).

1.1.1. \( h(u) \neq t(v) \) and \( t(u) \neq h(v) \). Then \( u \overset{1.4}{\approx} xyzwxyz \) and \( v \overset{1.4}{\approx} zwxyszw \) for some distinct letters \( x, y, z, w \in X \) and simple word \( s \in X^* \) with \( x, y, z, w \notin C(s) \). Denote by \( \varphi_1 \) the substitution
\[
v \mapsto \begin{cases} x & \text{if } v \in C(ys), \\ y & \text{if } v \in C(zw). \end{cases}
\]
Then
\[
\{(1, 4), \sigma \} \vdash xyx \overset{1.4}{\approx} u \varphi_1 \overset{1.4}{\approx} v \varphi_1 \overset{1.4}{\approx} yxy \vdash xyx \approx yxy.
\]
Denote by \( \chi_1 \) the substitution
\[
v \mapsto \begin{cases} xy & \text{if } v = x, \\ x & \text{if } v = y, \\ y & \text{if } v \in C(zs). \end{cases}
\]
Then
\[
\{(1, 4), xyx \approx yxy \} \vdash u \overset{1.4}{\approx} (xyx) \chi_1 \overset{1.4}{\approx} (yxy) \chi_1 \overset{1.4}{\approx} v \vdash \sigma.
\]
Hence \( \overline{B}_2(\sigma) = \overline{B}_2\{xyx \approx yxy\} \overset{4}{=} \overline{B}_2\{\mu_0, 0\} \).

1.1.2. \( h(u) = t(v) \) and \( t(u) \neq h(v) \). Then \( u \overset{1.4}{\approx} xzsxyz \) and \( v \overset{1.4}{\approx} zxszzx \) for some distinct letters \( x, y, z \in X \) and simple word \( s \in X^* \) with \( x, y, z \notin C(s) \). Denote by \( \varphi_2 \) the substitution
\[
v \mapsto \begin{cases} xy & \text{if } v = x, \\ y & \text{if } v = y, \\ z & \text{if } v \in C(zs). \end{cases}
\]
Then
\[
\{(1, 4), \sigma \} \vdash xyx \overset{1.4}{\approx} u \varphi_2 \overset{1.4}{\approx} v \varphi_2 \overset{1.4}{\approx} yxy \vdash xyx \approx yxy.
\]
Denote by \( \chi_2 \) the substitution
\[
v \mapsto \begin{cases} xy & \text{if } v = x, \\ z & \text{if } v = y. \end{cases}
\]
Then
\[
\{(1, 4), xyx \approx yxy \} \vdash u \overset{1.4}{\approx} (xyx) \chi_2 \overset{1.4}{\approx} (yxy) \chi_2 \overset{1.4}{\approx} v \vdash \sigma.
\]
Hence \( \overline{B}_2(\sigma) = \overline{B}_2\{xyx \approx yxy\} \overset{4}{=} \overline{B}_2\{\mu_0, 0\} \).
1.1.3. \( h(u) \neq t(v) \) and \( t(u) = h(v) \). This is symmetrical to Case 1.1.2.

1.1.4. \( h(u) = t(v) \) and \( t(u) = h(v) \). Then \( u \approx xysxz \) and \( v \approx yzsyx \) for some distinct letters \( x, y \in X \) and simple word \( s \in X^* \) with \( x, y \notin C(s) \). Denote by \( \varphi_3 \) the substitution

\[
\begin{align*}
v &\mapsto \begin{cases} 
xy & \text{if } v = x, 
\end{cases} 
\end{align*}
\]

Then

\[
\{(1, 4), \sigma\} \vdash xyx \approx yxy. 
\]

Denote by \( \chi_3 \) the substitution

\[
\begin{align*}
v &\mapsto \begin{cases} 
xsy & \text{if } v = x, 
yx & \text{if } v = y. 
\end{cases} 
\end{align*}
\]

Then

\[
\{(1, 4), xyx \approx yxy\} \vdash u \approx (xyx)\chi_3 \approx (yxy)\chi_3 \approx v \vdash \sigma. 
\]

Hence \( \mathcal{B}_2[\sigma] = \mathcal{B}_2\{xyx \approx yxy\} \approx \mathcal{B}_2\{\mu_{0,0}\} \).

1.2. \( h(u) = h(v) \) and \( t(u) \neq t(v) \). Then \( u \approx xysxz \) and \( v \approx yzsyx \) for some distinct letters \( x, y, z \in X \) and simple word \( s \in X^* \) with \( x, y, z \notin C(s) \). Denote by \( \varphi_4 \) the substitution \( v \mapsto x \) for all \( v \in C(zs) \). Then

\[
\{(1, 4), \sigma\} \vdash xyx \approx yxy. 
\]

Denote by \( \chi_4 \) the substitution \( x \mapsto xsz \). Then

\[
\{(1, 4), xyx \approx yxy\} \vdash u \approx (xyx)\chi_4 \approx (yxy)\chi_4 \approx v \vdash \sigma. 
\]

Hence \( \mathcal{B}_2[\sigma] = \mathcal{B}_2\{xyx \approx yxy\} \approx \mathcal{B}_2\{\rho_{0,0}\} \).

1.3. \( h(u) \neq h(v) \) and \( t(u) = t(v) \). It follows from an argument symmetrical to Case 1.2 that \( \mathcal{B}_2[\sigma] = \mathcal{B}_2\{\lambda_{0,0}\} \).

Case 2. \( u \) is of the form (B1) and \( v \) is of the form (B2).

2.1. \( h(u) \neq h(v) \) and \( t(u) \neq t(v) \). Then \( u \approx xyszx \) and \( v \approx yzxsz \) for some distinct letters \( x, y, z \in X \) and simple word \( s \in X^* \) with \( x, y, z \notin C(s) \). Denote by \( \varphi_5 \) the substitution \( v \mapsto y \) for all \( v \in C(zs) \). Then

\[
\{(1, 4), \sigma\} \vdash xyx \approx yzxsz \approx yxy. 
\]
Denote by $\chi_5$ the substitution $y \mapsto ysz$. Then
\begin{equation}
\{(1,4), xyx \approx yxy\} \vdash u \overset{(1,4)}{\approx} (xyx)\chi_5 \approx (yxy)\chi_5 \overset{(1,4)}{\approx} v \vdash \sigma.
\end{equation}
Hence $B_2(\sigma) = B_2\{xyx \approx yxy\} \overset{(4)}{=} B_2\{\mu_{0,0}\}.$

2.2. $h(u) = h(v)$ and $t(u) \neq t(v)$. Then $u \overset{(1,4)}{\approx} ysx$ and $v \overset{(1,4)}{\approx} ysx$ for some distinct letters $x, y \in \mathcal{X}$ and simple word $s \in \mathcal{X}^*$ with $x, y \notin C(s)$. By making the substitution $v \mapsto y$ for all $v \in C(s)$, it is easy to show that
\begin{equation}
\{(1,4), \sigma\} \vdash xyx \approx xyx.
\end{equation}
Denote by $\chi_6$ the substitution $y \mapsto sy$. Then
\begin{equation}
\{(1,4), xyx \approx xyx\} \vdash u \overset{(1,4)}{\approx} (xyx)\chi_6 \approx (xyx)\chi_6 \overset{(1,4)}{\approx} v \vdash \sigma.
\end{equation}
Hence $B_2(\sigma) = B_2\{xyx \approx xyx\} \overset{(4)}{=} B_2\{\rho_{0,0}\}.$

2.3. $h(u) \neq h(v)$ and $t(u) = t(v)$. It follows from an argument symmetrical to Case 2.2 that $B_2(\sigma) = B_2\{\lambda_{0,0}\}.$

Case 3. $u$ and $v$ are both of the form $(B1)$. Then $h(u) \neq h(v)$ and $t(u) \neq t(v)$ necessarily. Hence $u \overset{(1,4)}{\approx} ysx$ and $v \overset{(1,4)}{\approx} yrsy$ for some distinct letters $x, y \in \mathcal{X}$ and simple word $s \in \mathcal{X}^*$ with $x, y \notin C(s)$. By making the substitution $v \mapsto y$ for all $v \in C(s)$, it is easy to show that
\begin{equation}
\{(1,4), \sigma\} \vdash xyx \approx yxy.
\end{equation}
Denote by $\chi_7$ the substitution $x \mapsto xsx$. Then
\begin{equation}
\{(1,4), xyx \approx yxy\} \vdash u \overset{(1,4)}{\approx} (xyx)\chi_7 \approx (yxy)\chi_7 \overset{(1,4)}{\approx} v \vdash \sigma.
\end{equation}
Hence $B_2(\sigma) = B_2\{xyx \approx yxy\} \overset{(4)}{=} B_2\{\mu_{0,0}\}.$ \hfill \ensuremath{\Box}

\textbf{Lemma 3.8.} Let $u \approx v$ be any identity of $A_0$ where $u$ and $v$ are distinct connected words. Then $B_2\{u \approx v\} = B_2\{\Sigma\}$ where
\begin{equation}
\Sigma = \begin{cases}
\{\lambda_{0,0}, \rho_{0,0}\} & \text{if } h(u) \neq h(v) \text{ and } t(u) \neq t(v), \\
\{\lambda_{0,0}\} & \text{if } h(u) \neq h(v) \text{ and } t(u) = t(v), \\
\{\rho_{0,0}\} & \text{if } h(u) = h(v) \text{ and } t(u) \neq t(v).
\end{cases}
\end{equation}

\textbf{Proof.} This follows from Lemmas 3.4, 3.6 and 3.7. \hfill \ensuremath{\Box}
3.4. The finite basis property for varieties in $\mathcal{I}_1$

Lemma 3.9. The equation $\overline{\mathcal{A}}_2\{\Sigma\} = \overline{\mathcal{B}}_2\{\Sigma\} \lor B_2$ holds for any $\Sigma \subseteq \{\lambda_{0,0}, \rho_{0,0}\}$.

Proof. First assume that $\Sigma = \{\lambda_{0,0}, \rho_{0,0}\}$. By [15, Theorem 4.1], the variety $A_0$ is defined by the identities in $\{x^1 \approx x^2, xyx \approx yxy \approx yxy\}$, whence it is routine to verify that $\overline{\mathcal{B}}_2\{\mu_{0,0}\} = A_0$. Therefore

$$\overline{\mathcal{B}}_2\{\mu_{0,0}\} \lor B_2 = A_0 \lor B_2 = A_2\{\mu_{0,0}\} = \overline{\mathcal{A}}_2\{\mu_{0,0}\},$$

where the second equality holds because $A_2\{\mu_{0,0}\} \subseteq \overline{\mathcal{A}}_2$. Hence

$$\overline{\mathcal{A}}_2\{\Sigma\} = \overline{\mathcal{A}}_2\{\mu_{0,0}\} = \overline{\mathcal{B}}_2\{\mu_{0,0}\} \lor B_2 = \overline{\mathcal{B}}_2\{\Sigma\} \lor B_2,$$

where the first and third equalities hold by Lemma 3.6.

Now assume that $\Sigma = \{\lambda_{0,0}\}$. It is routine to show by Proposition 2.5(ii) and Lemma 3.3 that the identities in $\{(1), xyx \approx yxyx\}$ define the variety $\overline{\mathcal{B}}_2\{\lambda_{0,0}\}$; this variety is denoted in [15] by $R_C\{0\}$ and is generated by the semigroup $RC_0$ of order four (see [15, Theorem 4.2(ii)]). Hence

$$\overline{\mathcal{B}}_2\{\lambda_{0,0}\} \lor B_2 = R_C\{0\} \lor B_2 = A_2\{\lambda_{0,0}\} = \overline{\mathcal{A}}_2\{\lambda_{0,0}\},$$

where the second equality holds by [15, Theorem 4.3(iii)] and the third equality holds because $A_2\{\lambda_{0,0}\} \subseteq \overline{\mathcal{A}}_2$.

The case when $\Sigma = \{\rho_{0,0}\}$ follows by a symmetrical argument. The case when $\Sigma = \emptyset$ is the main result of [34]. \qed

Lemma 3.10. Let $u \approx v$ be any identity where

(a) $u$ and $v$ are connected words such that either $h(u) \neq h(v)$ or $t(u) \neq t(v)$;
(b) $A_0 \lor B_2 \vdash u \approx v$.

Then $\overline{\mathcal{A}}_2\{u \approx v\} = \overline{\mathcal{A}}_2\{\Sigma\}$ where

$$\Sigma = \begin{cases} \{\lambda_{0,0}, \rho_{0,0}\} & \text{if } h(u) \neq h(v) \text{ and } t(u) \neq t(v), \\ \{\lambda_{0,0}\} & \text{if } h(u) \neq h(v) \text{ and } t(u) = t(v), \\ \{\rho_{0,0}\} & \text{if } h(u) = h(v) \text{ and } t(u) \neq t(v). \end{cases}$$

Proof. Suppose that $h(u) \neq h(v)$. Denote by $\varphi$ the substitution

$$z \mapsto \begin{cases} x^2 & \text{if } z = h(u), \\ y^2 & \text{otherwise}. \end{cases}$$

Then

$$\{(1), u \approx v\} \vdash x^2y^2x^2 \approx (u\varphi)x^2 \approx (v\varphi)x^2 \approx y^2x^2y^2x^2 \vdash \lambda_{0,0}. \quad \text{(1)}$$

By symmetry, if $t(u) \neq t(v)$, then the deduction $\{(1), u \approx v\} \vdash \rho_{0,0}$ holds. Therefore the inclusion $\overline{\mathcal{A}}_2\{u \approx v\} \subseteq \overline{\mathcal{A}}_2\{\Sigma\}$ holds.
Conversely, since $\overline{B}_2\{u \approx v\} = \overline{B}_2\{\Sigma\}$ by Lemma 3.8,

$$\overline{A}_2\{\Sigma\} = \overline{B}_2\{\Sigma\} \lor B_2 = \overline{B}_2\{u \approx v\} \lor B_2 \subseteq \overline{A}_2\{u \approx v\}$$

where the first equality holds by Lemma 3.9.

**Lemma 3.11.** The variety $\overline{A}_2$ satisfies the identities

$$\begin{align*}
\{xhy^2z^2tx \approx xhz^2y^2tx, & \quad xy^2z^2x \approx xz^2y^2x, \\
\{xhy^2z^2x \approx xhz^2y^2x, & \quad xy^2z^2tx \approx xz^2y^2tx.\}
\end{align*}$$

**(5)**

**Proof.** The identities in (5) hold in $\overline{B}_2$ by Lemma 3.3, and they hold in $B_2$ since the idempotents of $B_2$ commute. The lemma now follows since $\overline{A}_2 = \overline{B}_2 \lor B_2$ by Lemma 3.9.

One important property of the identities in (5) is their ability to permute any two adjacent square factors of a word that are sandwiched between two occurrences of a non-simple letter. More generally, if $x, y, z, w \in X^+$ and $h, t \in X^*$ are such that $C(x) \cap C(w) \neq \emptyset$, then

$$xhy^2z^2tw \overset{(5)}{=} xhz^2y^2tw.$$

This property will be required to prove the following lemma.

**Lemma 3.12.** Let $\sigma : a_\ell(\ell)ub_\ell(\ell) \approx a_\ell(\ell)vb_\ell(\ell)$ be any identity where

(a) $u$ and $v$ are connected words such that $h(u) \neq h(v)$ and $t(u) \neq t(v)$;

(b) $A_0 \lor B_2 \models u \approx v$.

Then $\overline{A}_2\{\sigma\} = \overline{A}_2\{\lambda_{\ell,r}, \rho_{\ell,r}\}$.

**Proof.** Denote by $\varphi$ the substitution

$$z \mapsto \begin{cases} x^2 & \text{if } z \in C(b(\ell)) \text{ or } z = h(u), \\
y^2 & \text{if } z \in C(u) \text{ and } z \neq h(u). \end{cases}$$

Then the inclusion $\overline{A}_2\{\sigma\} \subseteq \overline{A}_2\{\lambda_{\ell,r}\}$ holds since

$$\{(1), \sigma\} \vdash a_\ell(\ell)x^2y^2z^2b_\ell(\ell) \overset{(1)}{=} ((a_\ell(\ell) ub_\ell(\ell))\varphi)x^2b_\ell(\ell) \overset{\approx}{=} ((a_\ell(\ell) vb_\ell(\ell))\varphi)x^2b_\ell(\ell) \overset{(1)}{=} a_\ell(\ell)y^2x^2y^2z^2b_\ell(\ell).$$

The inclusion $\overline{A}_2\{\sigma\} \subseteq \overline{A}_2\{\rho_{\ell,r}\}$ also holds by a symmetrical argument. Therefore $\overline{A}_2\{\sigma\} \subseteq \overline{A}_2\{\lambda_{\ell,r}, \rho_{\ell,r}\}$ and it remains to verify the reverse inclusion. By Lemma 3.6, it suffices to show that $\overline{A}_2\{\mu_{\ell,r}\} \subseteq \overline{A}_2\{\sigma\}$.
Recall from Lemma 2.2 and Proposition 2.5(i) that the variety $\mathcal{A}_2\{\mu_{0,0}\}$ is defined by the identities
\begin{equation}
(1), \quad x^2y^2x^2 \approx x^2yx^2, \quad \mu_{0,0}.
\end{equation}
Since $\mathcal{A}_2\{u \approx v\} = \mathcal{A}_2\{\mu_{0,0}\}$ by Lemmas 3.6 and 3.10, there exists a deduction sequence
\begin{equation}
u = w^{(0)} \Rightarrow w^{(1)} \Rightarrow \cdots \Rightarrow w^{(s)} = v
\end{equation}
where each deduction $w^{(j)} \Rightarrow w^{(j+1)}$ involves that there exist words $e_j, f_j \in \mathcal{X}^*$, an identity $g_j = h_j$ from (6), and an endomorphism $\gamma_j$ of $\mathcal{X}^+$ such that
\begin{equation}w^{(j)} = e_j(g_j\gamma_j)f_j \quad \text{and} \quad w^{(j+1)} = e_j(h_j\gamma_j)f_j.
\end{equation}
Since the word $u$ is connected and $\mathcal{A}_0 \models u \approx w^{(j)}$ for all $j$, each $w^{(j)}$ is also a connected word by Lemma 3.1(i).

In each of the following cases, it is shown that each deduction $w^{(j)} \Rightarrow w^{(j+1)}$ in (7) implies a deduction sequence $a_{(t)} w^{(j)} b_{(r)} \Rightarrow \cdots \Rightarrow a_{(t)} w^{(j+1)} b_{(r)}$ that involves the identities
\begin{equation}(1,5), \quad x^2y^2x^2 \approx x^2yx^2, \quad \mu_{\ell,r}
\end{equation}
of $\mathcal{A}_2\{\mu_{\ell,r}\}$. Consequently, there exists a deduction sequence
\begin{equation}a_{(t)} ub_{(r)} = a_{(t)} w^{(0)} b_{(r)} \Rightarrow \cdots \Rightarrow a_{(t)} w^{(s)} b_{(r)} = a_{(t)} vb_{(r)}
\end{equation}
that involves identities of $\mathcal{A}_2\{\mu_{\ell,r}\}$, whence $\mathcal{A}_2\{\mu_{\ell,r}\} \subseteq \mathcal{A}_2\{\sigma\}$ as required.

If the deduction $w^{(j)} \Rightarrow w^{(j+1)}$ involves the application of an identity from (6) that is not $\mu_{0,0}$, then clearly the deduction $a_{(t)} w^{(j)} b_{(r)} \Rightarrow a_{(t)} w^{(j+1)} b_{(r)}$ can be achieved by the identities in (8). Hence suppose that the deduction $w^{(j)} \Rightarrow w^{(j+1)}$ involves an application of the identity $\mu_{0,0}$. Then $w^{(j)} = e_j((x^2y^2x^2)\gamma_j) f_j$ and $w^{(j+1)} = e_j((x^2y^2\gamma_j) f_j)$ so that
\begin{align*}
a_{(t)} w^{(j)} b_{(r)} &= a_{(t)} e_j c^2 d^2 e^2 c^2 f_j b_{(r)}, \\
a_{(t)} w^{(j+1)} b_{(r)} &= a_{(t)} e_j d^2 c^2 d^2 f_j b_{(r)},
\end{align*}
where $c = x\gamma_j$ and $d = y\gamma_j$. There are four cases.

Case 1. $e_j = \emptyset = f_j$. The deduction $a_{(t)} w^{(j)} b_{(r)} \Rightarrow a_{(t)} w^{(j+1)} b_{(r)}$ clearly requires only a single application of the identity $\mu_{\ell,r}$ in (8).

Case 2. $e_j = \emptyset \neq f_j$. Since $w^{(j)} = c^2 d^2 c^2 f_j$ is connected, either $C(c) \cap C(f_j)$ or $C(d) \cap C(f_j)$ is nonempty. In both cases,
\begin{equation}a_{(t)} d c^2 d^2 c^2 \cdot f_j b_{(r)} \approx a_{(t)} d^2 c^2 d^2 c^2 d^2 f_j b_{(r)} \approx a_{(t)} w^{(j+1)} b_{(r)}.
\end{equation}
It is easy to show that \( \{(1), \mu_{\ell,r}\} \vdash \mu_{\ell,\omega+1} \) so that

\[
\{(1, 5), \mu_{\ell,r}\} \vdash a(t)w^{(j)}b_{(r)} \overset{(1)}{=} (a(t)c^2d^2c^2)j_f b_{(r)} \\
\overset{\mu_{\ell,\omega+1}}{=} a(t)d^2c^2d^2f_j b_{(r)} \\
\overset{(1, 5)}{=} a(t)w^{(j+1)}b_{(r)}.
\]

Therefore the deduction \( a(t)w^{(j)}b_{(r)} \Rightarrow a(t)w^{(j+1)}b_{(r)} \) can be achieved by applications of the identities in (8).

**Case 3.** \( e_j \neq \emptyset = f_j \). This is symmetrical to Case 2.

**Case 4.** \( e_j \neq \emptyset \neq f_j \).

*4.1.** \( C(e_j) \cap C(f_j) \neq \emptyset \). Then

\[
a(t)w^{(j)}b_{(r)} \overset{(1)}{=} a(t)(e_j c^2d^2c^2f_j)j_f b_{(r)} \\
\overset{(5)}{=} a(t)e_j d^2c^2d^2f_j b_{(r)} \\
\overset{(1)}{=} a(t)w^{(j+1)}b_{(r)}.
\]

*4.2.** \( C(e_j) \cap C(f_j) = \emptyset \). Since \( w^{(j)} = e_j c^2d^2c^2f_j \) is connected, either \( C(e_j) \cap C(c) \) or \( C(e_j) \cap C(d) \) is nonempty, and either \( C(c) \cap C(f_j) \) or \( C(d) \cap C(f_j) \) is nonempty.

*4.2.1.** \( C(e_j) \cap C(c) \neq \emptyset \neq C(c) \cap C(f_j) \). Then

\[
a(t)w^{(j)}b_{(r)} \overset{(1)}{=} a(t)(e_j c^2d^2d^2c^2)j_f b_{(r)} \\
\overset{(5)}{=} a(t)e_j d^2c^2d^2c^2f_j b_{(r)} \\
\overset{(5)}{=} a(t)e_j d^2c^2d^2f_j b_{(r)} \\
\overset{(1)}{=} a(t)w^{(j+1)}b_{(r)}.
\]

*4.2.2.** \( C(e_j) \cap C(c) \neq \emptyset \neq C(d) \cap C(f_j) \). Then

\[
a(t)w^{(j)}b_{(r)} \overset{(1)}{=} a(t)(e_j c^2d^2c^2f_j)j_f b_{(r)} \\
\overset{(5)}{=} a(t)e_j d^2c^2d^2c^2f_j b_{(r)} \\
\overset{(5)}{=} a(t)e_j d^2c^2d^2f_j b_{(r)} \\
\overset{(1)}{=} a(t)w^{(j+1)}b_{(r)}.
\]

*4.2.3.** \( C(e_j) \cap C(d) \neq \emptyset \neq C(c) \cap C(f_j) \). This is symmetrical to Case 4.2.2.

*4.2.4.** \( C(e_j) \cap C(d) \neq \emptyset \neq C(d) \cap C(f_j) \). This is Case 4.2.1 with the words \( c \) and \( d \) interchanged. □
Lemma 3.13. Let \( \sigma : a^{(\ell)}ub^{(r)} \approx a^{(\ell)}vb^{(r)} \) be any identity where

(a) \( u \) and \( v \) are connected words such that either \( h(u) \neq h(v) \) or \( t(u) \neq t(v) \);
(b) \( A_0 \lor B_2 \models u \approx v \).

Then \( \overline{A}_2\{\sigma\} = \overline{A}_2\{\Sigma\} \) where

\[
\Sigma = \{ \lambda_{\ell,r}, \rho_{\ell,r} \} \quad \text{if} \quad h(u) \neq h(v) \text{ and } t(u) \neq t(v),
\]

\[
\{ \rho_{\ell,r} \} \quad \text{if} \quad h(u) = h(v) \text{ and } t(u) \neq t(v).
\]

Proof. The case when \( h(u) \neq h(v) \) and \( t(u) \neq t(v) \) is Lemma 3.12. The other cases can be proved in a very similar manner. \( \square \)

Proposition 3.14. Each variety in the interval \( \mathcal{I}_1 = [A_0 \lor B_2, \overline{A}_2] \) is finitely based.

Proof. Suppose that \( V \in \mathcal{I}_1 \). Then it follows from Lemma 3.2 that \( V = \overline{A}_2\{\Sigma\} \) for some set \( \Sigma \) of identities of \( A_0 \lor B_2 \) of the form (3) that satisfy conditions (a) to (e) in that lemma. Consider any identity \( \sigma : a^{(\ell)}ub^{(r)} \approx a^{(\ell)}vb^{(r)} \) in \( \Sigma \) with \( h(u) = h(v) \) and \( t(u) = t(v) \). Then the identity \( u \approx v \) holds in \( \overline{B}_2 \) by Lemma 3.3. Since \( \overline{B}_2 \lor B_2 = \overline{A}_2 \) by Lemma 3.9, the identity \( u \approx v \) also holds in \( \overline{A}_2 \). It follows that the identity \( \sigma \) can be omitted from \( \Sigma \) without affecting the definition of \( V \), that is, \( V = \overline{A}_2\{\Sigma \setminus \{\sigma\}\} \). Therefore every identity in \( \Sigma \) can be assumed to be of the form (3) that satisfies condition (a) in Lemma 3.13; condition (b) in Lemma 3.13 is an assumption of the present lemma so that \( \Sigma \) can in fact be chosen to be a subset of \( \{ \lambda_{\ell,r}, \rho_{\ell,r} : \ell, r \in \mathbb{N}^* \} \). Now since the identities in \( \Sigma \) are formed by words of level at most eight, the variety defined by \( \Sigma \) is finitely based by Lemma 2.1. Consequently, the variety \( V = \overline{A}_2\{\Sigma\} \) is also finitely based by Proposition 2.5. \( \square \)

4. The Intervals \( \mathcal{I}_3, \mathcal{I}_4, \) and \( \mathcal{I}_5 \)

4.1. The finite basis property for varieties in \( \mathcal{I}_3 \) and \( \mathcal{I}_4 \)

Recall from Proposition 2.5(iii) that the identity

\[
x^2y^2x^2y^2 \approx x^2y^2,
\]

together with the identities (1), constitute a basis for \( A_0 \).

Lemma 4.1. Suppose that \( w = w_1 \cdots w_m \) where \( w_1, \ldots, w_m \) are disjoint connected words. Then the deduction (1, 9) \( \vdash \sigma \approx w^c \) holds for some connected word \( w^c \) with \( C(w) = C(w^c) \).

Proof. It suffices to verify the lemma for \( m = 2 \) as the general case can be obtained by induction. Since \( w_1 \) is a connected word, the letter \( t = t(u) \) occurs at least twice
in \(w_1\). Therefore \(w_1 = w_1^t w_1^r t\) for some \(w_1^t, w_1^r \in X^+\), whence \(w_1 \equiv w_1(w_1^r t)^2\). Thus \(w_1 \equiv w_1 e^2\) for some \(e \in X^+\) with \(t(e) = t(w_1)\). Similarly, the letter \(h(w_2)\) occurs at least twice in \(w_2\) so that \(w_2 \equiv f^2 w_2\) for some \(f \in X^+\) with \(h(f) = h(w_2)\). Consequently, \(w_1 w_2 \equiv w_1 e^2 f^2 w_2 \equiv w_1 e^2 f^2 \approx w_1 f^2 e^2 w_2\) where \(w^c = w_1 f^2 e^2 w_2\) is a connected word with \(C(w^c) = C(w_1 w_2)\).

Lemma 4.2. Let \(u \equiv v\) be any identity of the semigroup \(B_0\) (respectively, \(B_2\)). Suppose that \(u = u_1 s u_2\) for some disjoint words \(u_1, s, u_2\) such that \(s\) is simple. Then \(v = v_1 s v_2\) for some words \(v_1\) and \(v_2\) such that \(v_1, s, \text{ and } v_2\) are disjoint. Further, \(u_1 \equiv v_1\) and \(u_2 \equiv v_2\) are identities of \(B_0\) (respectively, \(B_2\)).

Proof. This follows from [15, Proposition 3.2(ii)] and its dual result.

Proposition 4.3. Each variety in the interval \(I_3 = [B_2, A_0]\) is finitely based.

Proof. Suppose that \(V \in I_3\). Then \(V = A_0\{\Sigma\}\) for some set \(\Sigma\) of nontrivial identities of \(B_2\). Consider any identity \(\sigma : u \equiv v\) in \(\Sigma\). If either \(u\) or \(v\) is simple, then it follows from Lemma 4.2 that \(u\) and \(v\) are contradictorily identical. Therefore both \(u\) and \(v\) are non-simple words, whence \(u = s_1 u_1 s_2 u_2 \cdots s_k u_k s_{k+1}\) where

- (a) \(s_1, \ldots, s_{k+1}\) are simple words with either \(s_1\) or \(s_{k+1}\) possibly being empty;
- (b) each \(u_i\) is either connected or a product of connected words;
- (c) \(s_1, u_1, \ldots, s_k, u_k, s_{k+1}\) are disjoint.

It follows from Lemma 4.2 that \(v = s_1 v_1 s_2 v_2 \cdots s_k v_k s_{k+1}\) where

- (d) each \(v_i\) is either connected or a product of connected words;
- (e) \(s_1, v_1, \ldots, s_k, v_k, s_{k+1}\) are disjoint;
- (f) the identities \(u_i \equiv v_i\) hold in the variety \(B_2\) (so that \(C(u_i) = C(v_i)\)).

By Lemma 4.1, there exist connected words \(u^c_i\) and \(v^c_i\) with \(C(u_i) = C(u^c_i)\) and \(C(v_i) = C(v^c_i)\) such that \((1, 9) \vdash \{u_i \equiv u^c_i, v_i \equiv v^c_i\}\). Since \(B_2\) satisfies the identities \((1, 9)\), it also satisfies the identities \(u^c_i \equiv v^c_i\). It follows from the proof of Part 4 of the first proposition in [4] that any two connected words with identical content form an identity for the semigroup \(A_0\). Hence the identities \(u^c_i \equiv v^c_i\) hold in \(A_0\). It follows that the identity \(\sigma' : u' \equiv v'\), where

\[
    u' = s_1 u^c_1 s_2 u^c_2 \cdots s_k u^c_k s_{k+1} \quad \text{and} \quad v' = s_1 v^c_1 s_2 v^c_2 \cdots s_k v^c_k s_{k+1},
\]

holds in \(A_0 \lor B_2\). The identities \((1, 9)\) of \(A_0\) were used to construct \(\sigma'\) from \(\sigma\) so that \(\overline{A_0}\{\sigma\} = \overline{A_0}\{\sigma'\}\).

Since \(\sigma\) is an arbitrarily chosen identity from \(\Sigma\), the construction of \(\sigma'\) from \(\sigma\) in the preceding paragraph can be repeated on every other identity from \(\Sigma\) to
obtain the set \( \Sigma' = \{ \sigma' : \sigma \in \Sigma \} \) of identities of \( A_0 \lor B_2 \) with the property that \( \overline{A}_0{\{\Sigma'} \} = \overline{A}_0{\{\Sigma} \} = V \). The variety \( \overline{A}_2{\{\Sigma'} \} \) belongs to the interval \( \mathcal{I}_1 \) and so is finitely based by Proposition 3.14. Since \( \overline{A}_0 \) is finitely based by Proposition 2.5, the variety \( V = \overline{A}_2{\{\Sigma'} \} \cap \overline{A}_0 \) is also finitely based.

**Proposition 4.4.** Each variety in the interval \( \mathcal{I}_4 = [B_0, \overline{A}_0 \cap \overline{B}_2] \) is finitely based.

**Proof.** The argument here is identical to the proof of Proposition 4.3. \( \square \)

**4.2. The finite basis property for varieties in \( \mathcal{I}_5 \)**

The finite basis property for all varieties in \( \mathcal{I}_5 \) has previously been established in [12]. Since the verification of this result is not long, the details are included for completeness.

Recall from Proposition 2.5(iv) that the identity

\[
 x^2 y^2 z^2 \approx x^2 yz^2 
\]  

holds in the variety \( B_0 \).

**Lemma 4.5.** Let \( w \) be a non-simple word. Then the deduction \( (1,10) \vdash w \approx s_1 w' s_2 \) holds for some \( s_1, w' s_2 \in X^+ \) such that

(a) \( s_1, s_2 \in X^+ \) are simple;
(b) \( w' \) is a \((4)\)-word;
(c) \( s_1, w', s_2 \) are disjoint.

**Proof.** Since \( w \) is a non-simple word, it can be written as \( w = s_1 u s_2 \) where \( s_1 \) is the longest simple prefix and \( s_2 \) is the longest simple suffix such that the factors \( s_1, u, \) and \( s_2 \) are disjoint. The letters \( h = h(u) \) and \( t = t(u) \) each occurs at least twice in \( u \) so that

\[
 u \overset{(1,4)}{\approx} h^2 u h^2 \overset{(10)}{\approx} h^2 u t^2 \overset{(1,4)}{\approx} u^2. 
\]

The word \( u^2 \) is clearly connected so that by Lemma 3.4, there exists some \((4)\)-word, say \( w' \), such that \( (1,4) \vdash u^2 \approx w' \). By Lemma 2.6, the identities \((4)\) hold in \( \overline{B}_0 \) so that \( (1,10) \vdash (4) \). Therefore

\[
 (1,10) \vdash w = s_1 u s_2 \overset{(1,4,10)}{\approx} s_1 u^2 s_2 \overset{(1,4)}{\approx} s_1 w' s_2 \vdash w \approx s_1 w' s_2
\]

as required. \( \square \)

**Proposition 4.6.** Each variety in the interval \( \mathcal{I}_5 = L(B_0) \) is finitely based.

**Proof.** Suppose that \( V \in \mathcal{I}_5 \). Then \( V = \overline{B}_0{\{\Sigma} \) for some set \( \Sigma \) of identities. Since the identities \((1,10)\) hold in the variety \( \overline{B}_0 \), it follows from Lemma 4.5 that the identities in \( \Sigma \) can be chosen so that they are formed by words that are either simple or non-simple of the form \( s_1 w' s_2 \). Therefore the identities in \( \Sigma \) are formed...
by words of level at most two, whence the variety defined by $\Sigma$ is finitely based by Lemma 2.1. Since the variety $\mathcal{B}_0$ is finitely based by Proposition 2.5, the variety $V = \mathcal{B}_0\{\Sigma\}$ is also finitely based.

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References


