FINITELY GENERATED LIMIT VARIETIES OF APERIODIC MONOIDS WITH CENTRAL IDEMPOTENTS

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A non-finitely based variety of algebras is said to be a limit variety if all its proper sub-varieties are finitely based. Recently, Marcel Jackson published two examples of finitely generated limit varieties of aperiodic monoids with central idempotents and questioned whether or not they are unique. The present article answers this question affirmatively.

Keywords: Varieties; limit varieties; monoids; aperiodic monoids; central idempotents; finitely based.

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1. Introduction

A variety of algebras is finitely based if its identities are finitely axiomatizable. A variety that is minimal with respect to being non-finitely based is said to be a limit variety. It follows from Zorn’s lemma that each non-finitely based variety contains some limit subvariety. Consequently, a classification of limit varieties within a certain class of varieties is, in a sense, equivalent to a description of varieties in the class that contain only finitely based subvarieties. Up to the present, explicit examples of limit varieties of groupoids [1, 5, 8], semigroups [9, 13, 14, 17], inverse semigroups [6], and bigroups [2] have been found. There are even examples of limit varieties that are minimal varieties [11, 16].

Recall that a monoid is aperiodic if all its subgroups are trivial. All varieties in this article are varieties of aperiodic monoids. Denote by \( \mathbf{A} \) the class of aperiodic monoids with central idempotents. Recently, Jackson [4] published two examples of finitely generated limit subvarieties of \( \mathbf{A} \); denote these two limit varieties by \( \mathbf{L}_1 \) and \( \mathbf{L}_2 \) (see Sec. 3 for more details). Not only are \( \mathbf{L}_1 \) and \( \mathbf{L}_2 \) the first published examples of limit varieties of monoids, they remain the only known explicit examples up to
the present. As commented by Jackson in [4, Sec. 5], no other examples of finitely
generated limit subvarieties of $A$ could be found. This led him to question the
uniqueness of $L_1$ and $L_2$ within $A$ [4, Question 1]. The main aim of the present
article is to answer this question.

**Theorem 1.1.** The only finitely generated limit subvarieties of $A$ are $L_1$ and $L_2$.

The proof of Theorem 1.1 is given in Sec. 3. It requires the finite basis property of
some subvarieties of $A$ which is established in Sec. 4.

A next step in the study of limit varieties of aperiodic monoids is to locate a
finitely generated example that is not contained in the class $A$. However, a com-
plete classification of limit varieties of monoids is feasible only within the class of
aperiodic monoids, since it follows from a result of Kozhevnikov [7] that there exist
uncountably many limit varieties of monoids consisting of groups.

2. Preliminaries

Denote by $X^+$ and $X^*$ respectively the free semigroup and free monoid over a
countably infinite alphabet $X$. Elements of $X$ are referred to as letters and elements
of $X^+$ and $X^*$ are referred to as words. Let $x$ be a letter and $w$ be a word. The
content of $w$ is the set of letters occurring in it and is denoted by $C(w)$. The
multiplicity of $x$ in $w$ is the number of times $x$ occurs in $w$ and is denoted by
$m(x, w)$. The letter $x$ is simple in $w$ if $m(x, w) = 1$. A word is simple if all its
letters are simple in it.

The index of an aperiodic semigroup $S$ is the least positive integer $n$ such that
the identity $x^n + 1 \approx x^n$ holds in $S$.

**Lemma 2.1.** Let $u \approx v$ be any identity such that $C(u) \neq C(v)$. Suppose that $u \approx v$
holds in some variety $V$ of aperiodic monoids of index $n$. Then $V$ is the variety of
trivial monoids.

**Proof.** By assumption, some letter $x$ occurs in either $u$ or $v$ but not both. Then the
identity $x^m \approx 1$ holds in $V$ for some $m \geq 1$. Since the monoids in $V$ are aperiodic
of index $n$, the identity $x^{n+1} \approx x^n$ holds in $V$. It follows that the identities $x^{mn} \approx 1$
and $x^{mn+1} \approx x^{mn}$ also hold in $V$. Therefore $V$ satisfies $1 \approx x^{mn} \approx x^{mn}x \approx x$ and
is trivial. \[\square\]

In the presence of Lemma 2.1 and since this article is only concerned with
varieties of aperiodic monoids with finite index, all identities in this article are
homotypical identities, that is, identities of the form $u \approx v$ with $C(u) = C(v)$. Let
$u \approx v$ and $u' \approx v'$ be identities. The deducibility of $u' \approx v'$ from $u \approx v$ within the
equational theory of monoids is indicated by $u \approx v \vdash u' \approx v'$. In particular, if $X_0$
is some subset of $X$ and $\alpha$ is the substitution $x \mapsto 1$ for all $x \in X_0$ (where 1 is the
identity element of the monoid $X^*$), then the deduction $u \approx v \vdash u \alpha \approx v \alpha$ holds.
This observation extends easily to the following result.
Lemma 2.2. Suppose that $u, v, w \in X^+$ with $C(u) \cap C(w) = C(v) \cap C(w) = \emptyset$. Then the identities $u \approx v$ and $wu \approx wv$ define the same variety of monoids.

2.1. Some finitely based varieties of monoids

Lemma 2.3. Any variety of commutative monoids is finitely based.

Proof. This can either be found in [3] or be deduced from [10, Theorem 22].

Lemma 2.4. Any finitely generated variety of monoids that satisfies either the identity $xyx \approx x^2y$ or its dual is finitely based.

Proof. Any variety of semigroups that satisfies either the identity $xyx \approx x^2y$ or its dual is finitely based [12, Theorem 1]. The lemma follows since a finite monoid is finitely based within the class of all monoids if and only if it is finitely based within the class of all semigroups (see [18, Sec. 1]).

Lemma 2.5 ([4], Lemmas 4.4 and 4.5(ii)). The variety of monoids defined by the identities $x^3 \approx x^2$ and $xyx \approx x^2y \approx y^2x$ contains precisely four subvarieties all of which are finitely based.

2.2. Some identity systems

Three types of identity systems will be used throughout this article. The first two types of systems are defined for each $n \geq 3$:

\[
\triangle_n = \left\{ \begin{array}{l}
x e_0 \prod_{i=1}^r (h_i x^{e_i}) \approx x f_0 \prod_{i=1}^r (h_i x^{f_i}) \quad \left( \begin{array}{l}
r \geq 1, \\
e_i \geq 0 \text{ and } f_i \geq 1 \text{ for all } i \geq 0, \\
e_i \ne f_i \text{ for some } i \geq 0, \\
\sum_{i=0}^r e_i \leq \sum_{i=0}^r f_i \leq n
\end{array} \right) \\
x e_0 \prod_{i=1}^r (h_i x^{e_i}) \approx x f_0 \prod_{i=1}^r (h_i x^{f_i}) \ne_i \ne f_i \text{ for some } i \geq 0, \\
\sum_{i=0}^r e_i \leq \sum_{i=0}^r f_i \leq n
\end{array} \right\},
\]

\[
\diamondsuit_n = \left\{ \begin{array}{l}
x e_0 y f_0 \prod_{i=1}^r (h_i x^{e_i} y^{f_i}) \quad \left( \begin{array}{l}
r \geq 0, \\
e_0, f_0 \geq 1, \\
e_i, f_i \geq 0 \text{ for all } i > 0, \\
e_i + f_i \geq 1 \text{ for all } i > 0, \\
2 \leq \sum_{i=0}^r e_i, \sum_{i=0}^r f_i < n
\end{array} \right) \\
x e_0 y f_0 \prod_{i=1}^r (h_i x^{e_i} y^{f_i}) \ne_0 \ne f_0 \text{ for some } i \geq 0, \\
\sum_{i=0}^r e_i \leq \sum_{i=0}^r f_i \leq n
\end{array} \right\}.
\]

The third type of systems is defined for each $n \geq 1$:

\[
\star_n = \left\{ \begin{array}{l}
x^{n+1} \approx x^n, \quad x^n y \approx y x^n, \quad x^n \prod_{i=1}^n y_i \approx \prod_{i=1}^n (y_i x)
\end{array} \right\}.
\]

The three identities in $\star_n$ are individually denoted by $\star_n^1, \star_n^2, \text{ and } \star_n^3$ respectively.
3. Jackson’s Limit Varieties and Their Uniqueness within \( \mathcal{A} \)

For any finite subset \( \mathcal{W} \) of \( \mathcal{X}^{\ast} \), denote by \( S(\mathcal{W}) \) the Rees quotient monoid of \( \mathcal{X}^{\ast} \) over the ideal of all words that are not factors of any word in \( \mathcal{W} \). Equivalently, \( S(\mathcal{W}) \) can be viewed as the monoid that consists of every factor of every word in \( \mathcal{W} \), together with a zero element \( 0 \), with binary operation \( \cdot \) given by

\[
\begin{cases}
  uv & \text{if } uv \text{ is a factor of some word in } \mathcal{W}, \\
  0 & \text{otherwise}.
\end{cases}
\]

(Note that the empty factor is the identity element of the monoid \( S(\mathcal{W}) \).)

The limit varieties \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) introduced in Sec. 1 are generated by the monoids \( S(\{ xhxyty \}) \) and \( S(\{ xhytxy, xyhxty \}) \) respectively [4, Proposition 5.1].

3.1. Proof of Theorem 1.1

A word \( w \) is an isoterm for a variety \( \mathcal{V} \) if there is no nontrivial identity of \( \mathcal{V} \) of the form \( w \approx w' \).

Lemma 3.1 ([4], Lemma 3.3 and Proposition 4.2).

(i) A monoid \( S(\mathcal{W}) \) belongs to a variety \( \mathcal{V} \) of monoids if and only if every word in \( \mathcal{W} \) is an isoterm for \( \mathcal{V} \).

(ii) The monoid \( S(\{ xyz \}) \) belongs to every finitely generated non-finitely based subvariety of \( \mathcal{A} \).

For any distinct letters \( x \) and \( y \) of a word \( w \), the expression \( x \prec_w y \) indicates the condition that within \( w \), any occurrence of \( x \) precedes any occurrence of \( y \).

Lemma 3.2. Let \( \mathcal{V} \) be any finitely generated limit subvariety of \( \mathcal{A} \). Suppose that \( \mathcal{V} \neq \mathcal{L}_1 \) and \( \mathcal{V} \neq \mathcal{L}_2 \). Then \( \mathcal{V} \) satisfies the identity system

\[
xhxyty \approx xhyxty, \quad xhytxy \approx xhytyx
\]

(3.1)
or its dual.

(The result of this lemma was stated at the end of [4, Sec. 5]. A proof is included here for completeness.)

**Proof.** Since \( \mathcal{V} \) is non-finitely based, it follows from Lemma 3.1 that \( xyx \) is an isoterm for \( \mathcal{V} \). Further, it follows from Lemma 2.3 that \( \mathcal{V} \) is noncommutative. Since \( \mathcal{L}_1 \) cannot be a subvariety of \( \mathcal{V} \), it follows from Lemma 3.1(i) that \( xhxyty \) is not an isoterm for \( \mathcal{V} \). Therefore a nontrivial identity of the form \( xhxyty \approx w \) holds in \( \mathcal{V} \). The equalities \( m(x, w) = m(y, w) = 2 \) and \( m(h, w) = m(t, w) = 1 \) hold because \( xyx \) is an isoterm for \( \mathcal{V} \). It follows from the noncommutativity of \( \mathcal{V} \) that

\[(a) \ h \prec_w t.\]

By Lemma 2.4, neither the identity \( xyx \approx x^2y \) nor its dual holds in \( \mathcal{V} \) so that

\[(b) \ x \not\approx_w h, \ h \not\approx_w x, \ y \not\approx_w t, \text{ and } t \not\approx_w y.\]
It is then easy to deduce from (a) and (b) that \( w = xhytxy \) is the only possibility. Hence the first identity in (3.1) holds in \( V \).

Similarly, since \( \mathbf{L}_2 \) cannot be a subvariety of \( V \), it follows from Lemma 3.1(i) that either \( xhytxy \) or \( xyhxty \) is not an isoterm for \( V \). By symmetry, it suffices to assume that \( xhytxy \) is not an isoterm for \( V \), whence a nontrivial identity of the form \( xhytxy \approx w \) holds in \( V \). It follows from an argument similar to the one in part (i) that \( w = xhytxy \). Therefore the second identity in (3.1) holds in \( V \). \( \square \)

Let \( V \) be any finitely generated subvariety of \( \mathbf{A} \). By the main theorem in [15], the identities \( \bigstar_n \) hold in \( V \) for some \( n \geq 1 \). Suppose that \( V \) is a limit variety that is different from \( \mathbf{L}_1 \) and \( \mathbf{L}_2 \). Then it follows from Lemma 3.2 that \( V \) satisfies either the identity system (3.1) or its dual. This is impossible since it is shown in the next section that any variety of monoids that satisfies the identities in \( \bigstar_n \cup \{(3.1)\} \) must be finitely based (Proposition 4.1).

4. The Variety \( Z_n \) and Its Subvarieties

For each \( n \geq 1 \), let \( Z_n \) be the variety of monoids defined by \( \bigstar_n \cup \{(3.1)\} \). It is easy to see that every \( Z_n \) is a subvariety of \( \mathbf{A} \). The main result of this section concerns the finite basis property of every subvariety of every \( Z_n \).

Proposition 4.1. Every subvariety of every \( Z_n \) is finitely based.

The proof of this proposition is divided into several parts and presented in the subsections that follow. It follows from Lemma 2.5 that all subvarieties of \( Z_n \) are finitely based if \( n \in \{1, 2\} \). Therefore it suffices to assume that \( n \geq 3 \) throughout this section.

For any set \( \Psi \) of identities, denote by \( Z_n(\Psi) \) the subvariety of \( Z_n \) defined by \( \Psi \).

4.1. Stacked words and \( n \)-regular identities

Suppose that a word \( w \) can be written as

\[
w = w_0 \prod_{i=1}^{r} (x^{e_i}w_i)
\]

where \( w_0, \ldots, w_r \in X^* \), \( x \in X \), \( e_1, \ldots, e_r \geq 1 \), and \( r \geq 1 \) are such that

(S1) \( x \notin \bigcup_{i=0}^{r} C(w_i) \);

(S2) \( w_i \) contains some simple letter of \( w \) if \( 0 < i < r \).

Then the letter \( x \) is said to be stacked in \( w \). Note that any simple letter of a word is vacuously stacked. A word is stacked if each of its letters is stacked.
An identity \( u \approx v \) is said to be \( n \)-regular if it satisfies all of the following conditions:

(R1) \( u \) and \( v \) are stacked words;
(R2) \( m(x, u) = m(x, v) < n \) for all \( x \in X \) (so that \( u \) and \( v \) contain the same simple letters);
(R3) if \( x \) and \( y \) are distinct simple letters of both \( u \) and \( v \), then \( x \prec_u y \) if and only if \( x \prec_v y \).

**Lemma 4.2.** Suppose that \( w \) is any word. Then the deduction (3.1) \( \vdash w \approx w' \) holds for some stacked word \( w' \).

**Proof.** This follows from the observation that if \( w = pwxr \) for some \( p, r \in X^* \) and \( q \in X^+ \) such that every letter of \( q \) is non-simple in \( w \), then the deduction (3.1) \( \vdash w \approx pwxr \) holds.

**Lemma 4.3.** Suppose that \( u \approx v \) is any identity such that \( m(x, u), m(x, v) \geq n \) for some \( x \in X \). Then

\[
Z_n \{ u \approx v \} = Z_n \{ u' \approx v' \}
\]

where the words \( u' \) and \( v' \) are, respectively, obtained from \( u \) and \( v \) by removing all occurrences of \( x \).

**Proof.** The equation \( Z_n \{ u \approx v \} = Z_n \{ x^n u' \approx x^n v' \} \) holds since

\[
u \overset{s_1^2}{\approx} x^{m(x, u)} u' \overset{s_1^1}{\approx} x^n u' \quad \text{and} \quad v \overset{s_1^2}{\approx} x^{m(x, v)} v' \overset{s_1^1}{\approx} x^n v'.
\]

It then follows from Lemma 2.2 that \( Z_n \{ u \approx v \} = Z_n \{ u' \approx v' \} \).

**Lemma 4.4.** Suppose that \( u \approx v \) is any identity that satisfies conditions (R1) and (R3) but not condition (R2). Then either \( Z_n \{ u \approx v \} \) is commutative or

\[\begin{align*}
Z_n \{ u \approx v \} = Z_n \{ \Delta, u' \approx v' \}
\end{align*}\]

for some set \( \Delta \) of identities from \( \triangle_n \) and some \( n \)-regular identity \( u' \approx v' \).

**Proof.** If \( u \) and \( v \) do not contain the same simple letters, then

\[
\bigstar_n \cup \{ u \approx v \} \vdash \{ x_n^2, x^2 \approx x \} \vdash xy \approx yx
\]

so that \( Z_n \{ u \approx v \} \) is commutative. Therefore assume that \( u \) and \( v \) contain the same simple letters. Further, in the presence of \( \bigstar_n \) and Lemma 4.3, it can be assumed that \( m(x, u), m(x, v) \leq n \) and \( (m(x, u), m(x, v)) \neq (n, n) \) for all \( x \in X \).

Suppose that \( m(y, u) \neq m(y, v) \) if and only if \( y \in \{ x_1, \ldots, x_m \} \).

Let \( e = m(x_1, u) \) and \( f = m(x_1, v) \). Without loss of generality, assume that \( e < f \leq n \). Since \( v \) is a stacked word, \( v = v_0 \prod_{i=1}^n x_i^{f_i} v_i \) for some \( v_0, \ldots, v_q \in X^* \) such that \( x_1 \notin \bigcup_{i=0}^q C(v_i) \), \( \sum_{i=1}^q f_i = f \), and for each \( i \notin \{0, q\} \), the word \( v_i \)
contains some simple letter, say $h_i$, of $v$. Denote by $\alpha$ the substitution $t \mapsto 1$ for all $t \not\in \{x_1, h_1, \ldots, h_q\}$. Then

$$v\alpha = x_1^{f_1} \prod_{i=1}^{q-1} (h_i x_1^{f_{i+1}}).$$

Similarly, since $u$ is a stacked word, $u = u_0 \prod_{i=1}^P (x_1^{f_i} u_i)$ for some $u_0, \ldots, u_p \in X^*$ such that $x_1 \not\in \bigcup_{i=0}^P C(u_i)$ and $\sum_{i=1}^P e_i = e$. By assumption, the simple letters $h_1, \ldots, h_p$ of $v$ are also simple letters of $u$. Since $u \approx v$ satisfies condition (R3),

$$u\alpha = x_1^{d_1} \prod_{i=1}^{q-1} (h_i x_1^{d_{i+1}})$$

for some $d_1, \ldots, d_q \geq 0$ such that $\sum_{i=1}^q d_i = e$. It is easy to see that $\tau_1 : u\alpha \approx v\alpha$ is an identity from $\mathbf{A}_n$. Note then that $Z_n\{u \approx v\} = Z_n\{\tau_1, u \approx v\}$. Let $\beta$ be the endomorphism on $X^+$ determined by the substitution $h_i \mapsto v_i$ for all $i \not\in \{0, q\}$ and let $u^{(1)} = u$ and $v^{(1)} = v_0 \prod_{i=1}^q (x_1^{d_i} v_i)$. Then

$$v = v_0((v\alpha)/\beta) v_q \approx v_0((u\alpha)/\beta) v_q = v^{(1)}$$

so that

$$Z_n\{u \approx v\} = Z_n\{\tau_1, u^{(1)} \approx v^{(1)}\}.$$

Note that $v^{(1)}$ is obtained from $v$ by changing the exponents from $(f_1, \ldots, f_q)$ to $(d_1, \ldots, d_q)$ and so is a stacked word. Therefore the identity $u^{(1)} \approx v^{(1)}$ satisfies conditions (R1) and (R3) with $m(y, u^{(1)}) = m(y, v^{(1)})$ for all $y \not\in \{x_2, \ldots, x_m\}$.

Since $m(x_2, u^{(1)}) \neq m(x_2, v^{(1)})$, the argument in the preceding paragraph can be repeated on the letter $x_2$. Hence

$$Z_n\{u^{(1)} \approx v^{(1)}\} = Z_n\{\tau_2, u^{(2)} \approx v^{(2)}\}$$

for some identity $\tau_2$ from $\mathbf{A}_n$ and some identity $u^{(2)} \approx v^{(2)}$ that satisfies conditions (R1) and (R3) with $m(y, u^{(2)}) = m(y, v^{(2)})$ for all $y \not\in \{x_3, \ldots, x_m\}$. It is easy to see how this procedure can be repeated on $x_3, \ldots, x_m$ to obtain identities $\tau_3, \ldots, \tau_m$ from $\mathbf{A}_n$ and identities $u^{(3)} \approx v^{(3)}, \ldots, u^{(m)} \approx v^{(m)}$ that satisfy conditions (R1) and (R3) such that

$$Z_n\{u^{(i)} \approx v^{(i)}\} = Z_n\{\tau_{i+1}, u^{(i+1)} \approx v^{(i+1)}\}$$

and $m(y, u^{(i)}) = m(y, v^{(i)})$ for all $y \not\in \{x_{i+1}, \ldots, x_m\}$. Note that $u^{(m)} \approx v^{(m)}$ is an $n$-regular identity. The equation in (4.2) now holds with $\Delta = \{\tau_1, \ldots, \tau_m\}$, $u' = u^{(m)}$, and $v' = v^{(m)}$. \hfill \Box

**Corollary 4.5.** Suppose that $V$ is any noncommutative subvariety of $Z_n$. Then $V = Z_n\{\Delta, \mathcal{R}\}$ for some set $\Delta$ of identities from $\mathbf{A}_n$ and some set $\mathcal{R}$ of $n$-regular identities.
Proof. By assumption, $V = Z_n(\mathcal{R})$ for some set $\mathcal{R}$ of identities that satisfy condition (R3). By Lemma 4.2, the words forming the identities in $\mathcal{R}$ can be chosen to be stacked so that all identities in $\mathcal{R}$ satisfy condition (R1). Since $V$ is non-commutative, $Z_n[\rho]$ cannot be commutative for any $\rho \in \mathcal{R}$. Therefore the result follows from Lemma 4.4. □

4.2. Partition sequences for stacked letters

Consider the word $w$ in (4.1) with stacked letter $x$. The factor $x^{e_1}$ is called the primary $x$-stack of $w$ while for any $i > 1$, the factor $x^{e_i}$ is called a secondary $x$-stack of $w$. Generally, the factors $x^{e_1}, \ldots, x^{e_r}$ are called $x$-stacks, or simply stacks, of $w$. It follows from (S2) that for each $i \in \{1, \ldots, r-1\}$, there exists some letter $h_i$ of $w_i$ that is simple in $w$. Such a letter sequence $(h_1, \ldots, h_{r-1})$ is said to be a partition sequence for $x$ in $w$. Denote by $\text{Par}(x, w)$ the set of all partition sequences of $x$ in $w$. Note that if there is only one $x$-stack in $w$, then $\text{Par}(x, w) = \emptyset$. It is convenient to define $\text{Par}(y, w) = \emptyset$ for any letter $y \notin C(w)$.

Example 4.6. Consider the word

$$w = h_1 \cdot x^3 y^2 \cdot h_2 h_3 \cdot t z^5 x^2 t \cdot h_4 \cdot x \cdot h_5 h_6 \cdot y^2 t x^6 \cdot h_7 \cdot y \cdot h_8.$$ 

(a) The letters $h_1, \ldots, h_8$ are simple in $w$ and so are stacks of $w$. Hence

$$\text{Par}(h_1, w) = \cdots = \text{Par}(h_8, w) = \emptyset.$$

(b) The letter $x$ is stacked and

$$\text{Par}(x, w) = \{(h_2, h_4, h_5), (h_2, h_4, h_6), (h_3, h_4, h_5), (h_3, h_4, h_6)\}.$$

The $x$-stacks are $x^3$, $x^2$, $x$, and $x^6$, with $x^3$ being the primary $x$-stack.

(c) The letter $y$ is stacked and

$$\text{Par}(y, w) = \{(h_2, h_7), (h_3, h_7), (h_4, h_7), (h_5, h_7), (h_6, h_7)\}.$$

The $y$-stacks are $y^2$, $y^2$, and $y$. The factor $y^2$ following the primary $x$-stack is the primary $y$-stack.

(d) The letter $z$ is stacked. Since $z^5$ is the only $z$-stack, it is also the primary $z$-stack with $\text{Par}(z, w) = \emptyset$.

(e) The letter $t$ is not stacked since the letters separating the first two occurrences of $t$ are all non-simple. Consequently, the word obtained from $w$ by removing all occurrences of $t$ is stacked.

For any pair of stacked words $u$ and $v$, the expression $\text{Par}(u) = \text{Par}(v)$ indicates the condition that $\text{Par}(x, u) = \text{Par}(x, v)$ for all $x \in X$. The number of $x$-stacks in a stacked word $w$ is denoted by $\text{sta}(x, w)$. Note that if $x$ is a stacked letter in the words $u$ and $v$ with $\text{Par}(x, u) = \text{Par}(x, v)$, then $\text{sta}(x, u) = \text{sta}(x, v)$ necessarily.
Lemma 4.7. Let $u \approx v$ be any $n$-regular identity. Suppose that there exists some $x \in X$ such that $\text{Par}(x, u) \neq \text{Par}(x, v)$ and $\text{sta}(x, u) + \text{sta}(x, v) \geq 3$. Then

$$Z_n\{u \approx v\} = Z_n\{\tau, u' \approx v'\}$$

for some identity $\tau$ from $\mathfrak{A}_n$ and some $n$-regular identity $u' \approx v'$ such that

(a) $\text{Par}(y, u) = \text{Par}(y, u')$ and $\text{Par}(y, v) = \text{Par}(y, v')$ for all $y \neq x$;
(b) $2 \leq \text{sta}(x, u') + \text{sta}(x, v') < \text{sta}(x, u) + \text{sta}(x, v)$.

**Proof.** By assumption, the words $u$ and $v$ can be written as

$$u = u_0 \prod_{i=1}^{p} (x^{e_i} u_i) \quad \text{and} \quad v = v_0 \prod_{i=1}^{q} (x^{f_i} v_i)$$

where $x^{e_1}, \ldots, x^{e_p}$ are the $x$-stacks of $u$ and $x^{f_1}, \ldots, x^{f_q}$ are the $x$-stacks of $v$ (so that $p + q = \text{sta}(x, u) + \text{sta}(x, v) \geq 3$) such that $\sum_{i=1}^{p} e_i = \sum_{i=1}^{q} f_i < n$.

**Case 1.** $p < q$. Let $\tilde{h} = (h_1, \ldots, h_{q-1})$ be any partition sequence of $x$ in $v$ (that is, $h_i$ is a letter in $v_i$ that is simple in $v$). Denote by $\alpha$ the substitution $t \mapsto 1$ for all $t \notin \{x, h_1, \ldots, h_{q-1}\}$. Then $v_0 = x^{\tilde{h}} \prod_{i=1}^{q-1} (h_i x^{f_{i+1}})$. By condition (R2), the simple letters $h_1, \ldots, h_{q-1}$ of $v$ are also simple in $u$. Hence it follows from condition (R3) that

$$u_0 \alpha = u_0 \alpha \prod_{i=1}^{p} (x^{e_i} (u_i \alpha))$$

where $\prod_{i=0}^{p} (u_i \alpha) = \prod_{i=1}^{q-1} h_i$. It is clear that $\text{sta}(x, u_0 \alpha) \leq p$ and that the identity $\tau : u_0 \alpha \approx v_0 \alpha$ belongs to $\mathfrak{A}_n$. Therefore $Z_n\{u \approx v\} = Z_n\{\tau, u \approx v\}$. Let $\beta$ be the endomorphism on $X^+$ determined by the substitution $h_i \mapsto v_i$ for all $i \notin \{0, q\}$ and let $u' = u$ and $v' = v_0((u_0 \alpha)\beta)v_p$. Then

$$v = v_0((v_0 \alpha)\beta)v_p \cong v_0((u_0 \alpha)\beta)v_p = v'$$

so that

$$Z_n\{u \approx v\} = Z_n\{\tau, u' \approx v'\}.$$

Note that

$$v' = v_0((u_0 \alpha)\beta)v_p = v_0 (u_0 \alpha \beta) \left( \prod_{i=1}^{p} (x^{e_i} (u_i \alpha \beta)) \right) v_p$$

where $\prod_{i=0}^{p} (u_i \alpha \beta) = \prod_{i=1}^{q-1} (h_i \beta) = \prod_{i=1}^{q-1} v_i$. Therefore the word $v'$ is obtained from $v$ by rearrangement of the $x$-stacks so that for any $y \neq x$, the $y$-stacks of $v$ and $v'$ are the same. It follows that condition (a) holds and $u' \approx v'$ is $n$-regular with $\text{sta}(x, u'), \text{sta}(x, v') \geq 1$. Since

$$\text{sta}(x, u') + \text{sta}(x, v') = \text{sta}(x, u) + \text{sta}(x, v) \leq p + p < p + q,$$

condition (b) also holds.
Case 2. $p > q$. This is symmetrical to Case 1.

Case 3. $p = q$. It is convenient to let $r = p = q$. By symmetry, it suffices to assume that there exists a partition sequence $\bar{h} = (h_1, \ldots, h_{r-1})$ of $x$ in $v$ that is not a partition sequence of $x$ in $u$. Denote by $\alpha$ the substitution $t \mapsto 1$ for all $t \notin \{x, h_1, \ldots, h_{r-1}\}$. Then $v_0 = x^h \prod_{i=1}^{r-1} (h_i x^{i+1})$. By condition (R2), the simple letters $h_1, \ldots, h_{r-1}$ of $v$ are also simple in $u$. Hence it follows from condition (R3) that

$$u_\alpha = u_0 \alpha \prod_{i=1}^r (x^c(u_i \alpha))$$

where $\prod_{i=0}^r (u_i \alpha) = \prod_{i=1}^{r-1} h_i$. If

$$(u_0 \alpha, u_1 \alpha, \ldots, u_{r-1} \alpha, u_r \alpha) = (\emptyset, h_1, \ldots, h_{r-1}, \emptyset),$$

then $h_i \in C(u_i)$ for all $i \notin \{0, r\}$ so that $\bar{h}$ is a partition sequence for $x$ in $u$, contradicting the choice of $\bar{h}$. Therefore $u_i \alpha = \emptyset$ for some $i \notin \{0, r\}$, whence $\text{sta}(x, u_\alpha) < r$. It is then clear that the identity $\tau : u_\alpha \approx v_0$ belongs to $\Delta_n$ and that $Z_n\{u \approx v\} = Z_n\{\tau, u \approx v\}$. Let $\beta$ be the endomorphism on $X^+$ determined by the substitution $h_i \mapsto u_i$ for all $i \notin \{0, q\}$ and let $u' = u$ and $v' = v_0((u_\alpha)\beta)v_q$. It follows from an argument similar to Case 1 that

$$Z_n\{u \approx v\} = Z_n\{\tau, u' \approx v'\},$$

where $u' \approx v'$ is an $n$-regular identity that satisfies condition (a) and the condition that $\text{sta}(x, u'), \text{sta}(x, v') \geq 1$. Since

$$\text{sta}(x, u') + \text{sta}(x, v') = \text{sta}(x, u) + \text{sta}(x, u_\alpha) < r + r = p + q,$$

condition (b) also holds.

Lemma 4.8. Let $u \approx v$ be any $n$-regular identity. Suppose that $\text{Par}(u) \neq \text{Par}(v)$. Then

$$Z_n\{u \approx v\} = Z_n\{\triangle, u' \approx v'\}$$

(4.3)

for some set $\triangle$ of identities from $\Delta_n$ and some $n$-regular identity $u' \approx v'$ such that $\text{Par}(u') = \text{Par}(v')$.

Proof. Let $x_1, \ldots, x_m$ be all the letters such that $\text{Par}(x_i, u) \neq \text{Par}(x_i, v)$. It follows from condition (R2) that $\text{sta}(x_i, u), \text{sta}(x_i, v) \geq 1$ for all $i$. Further, if there exists some $i$ such that $\text{sta}(x_i, u) = \text{sta}(x_i, v) = 1$, then $\text{Par}(x_i, u) = \emptyset = \text{Par}(x_i, v)$ is a contradiction. Therefore $\text{sta}(x_i, u) + \text{sta}(x_i, v) \geq 3$ for all $i$.

Define $u^{(0)} = u$ and $v^{(0)} = v$. For each $j \geq 0$, if $\text{Par}(x_1, u^{(j)}) \neq \text{Par}(x_1, v^{(j)})$ and $\text{sta}(x_1, u^{(j)}) + \text{sta}(x_1, v^{(j)}) \geq 3$, then it follows from Lemma 4.7 that

$$Z_n\{u^{(j)} \approx v^{(j)}\} = Z_n\{\tau_{j+1}, u^{(j+1)} \approx v^{(j+1)}\}$$
for some identity $\tau_{j+1}$ from $\Delta_n$ and some $n$-regular identity $u^{(j+1)} \approx v^{(j+1)}$ such that

(a) $\text{Par}(y, u^{(j)}) = \text{Par}(y, u^{(j+1)})$ and $\text{Par}(y, v^{(j)}) = \text{Par}(y, v^{(j+1)})$ for all $y \neq x_1$;
(b) $2 \leq \text{sta}(x_1, u^{(j+1)}) + \text{sta}(x_1, v^{(j+1)}) < \text{sta}(x_1, u^{(j)}) + \text{sta}(x_1, v^{(j)})$.

Note that the $n$-regular identities $u^{(0)} \approx v^{(0)}, u^{(1)} \approx v^{(1)}, \ldots$ and the identities $\tau_1, \tau_2, \ldots$ from $\Delta_n$ have just been defined recursively. If $\text{Par}(x_1, u^{(j)}) \neq \text{Par}(x_1, v^{(j)})$ and $\text{sta}(x_1, u^{(j)}) + \text{sta}(x_1, v^{(j)}) \geq 3$ for every $j \geq 1$, then it follows from (b) that there exists an infinite decreasing sequence

$$\text{sta}(x_1, u^{(1)}) + \text{sta}(x_1, v^{(1)}) > \text{sta}(x_1, u^{(2)}) + \text{sta}(x_1, v^{(2)}) > \cdots \geq 2$$

of integers, which is impossible. Hence there exists some $r \geq 1$ such that either

(c) $\text{Par}(x_1, u^{(r)}) = \text{Par}(x_1, v^{(r)})$ or
(d) $\text{sta}(x_1, u^{(r)}) + \text{sta}(x_1, v^{(r)}) = 2$.

Note that (d) is equivalent to $\text{sta}(x_1, u^{(r)}) = \text{sta}(x_1, v^{(r)}) = 1$, which implies that $\text{Par}(x_1, u^{(r)}) = \emptyset = \text{Par}(x_1, v^{(r)})$. Hence (c) must hold. Let $u^{x_1} = u^{(r)}, v^{x_1} = v^{(r)}$, and $\Delta^{x_1} = \{\tau_1, \ldots, \tau_r\}$. Then

$$Z_n \{u \approx v\} = Z_n \{\Delta^{x_1}, u^{x_1} \approx v^{x_1}\}$$

where $\Delta^{x_1}$ is some set of identities from $\Delta_n$ and $u^{x_1} \approx v^{x_1}$ is some $n$-regular identity such that $\text{Par}(x_1, u^{x_1}) = \text{Par}(x_1, v^{x_1})$. Further, it follows from (a) that $\text{Par}(y, u) = \text{Par}(y, u^{x_1})$ and $\text{Par}(y, v) = \text{Par}(y, v^{x_1})$ for all $y \neq x_1$.

Since $\text{Par}(x_2, u^{x_1}) \neq \text{Par}(x_2, v^{x_1})$ and $\text{sta}(x_2, u^{x_1}) + \text{sta}(x_2, v^{x_1}) \geq 3$, the argument in the previous paragraph can be repeated on the letter $x_2$ of the $n$-regular identity $u^{x_1} \approx v^{x_1}$ so that

$$Z_n \{u^{x_1} \approx v^{x_1}\} = Z_n \{\Delta^{x_2}, u^{x_2} \approx v^{x_2}\}$$

where $\Delta^{x_2}$ is some set of identities from $\Delta_n$ and $u^{x_2} \approx v^{x_2}$ is some $n$-regular identity such that $\text{Par}(x_i, u^{x_2}) = \text{Par}(x_i, v^{x_2})$ for $i \in \{1, 2\}$. It is easy to see how this procedure can be repeated on $x_3, \ldots, x_m$ to obtain sets $\Delta^{x_3}, \ldots, \Delta^{x_m}$ of identities from $\Delta_n$ and $n$-regular identities $u^{x_3} \approx v^{x_3}, \ldots, u^{x_m} \approx v^{x_m}$ such that

$$Z_n \{u^{x_i} \approx v^{x_i}\} = Z_n \{\Delta^{x_{i+1}}, u^{x_{i+1}} \approx v^{x_{i+1}}\}$$

and $\text{Par}(x_i, u^{x_{i+1}}) = \text{Par}(x_i, v^{x_{i+1}})$ for all $i \in \{1, \ldots, j + 1\}$. Note that $u^{x_m} \approx v^{x_m}$ is an $n$-regular identity such that $\text{Par}(u^{x_m}) = \text{Par}(v^{x_m})$. The equation in (4.3) now holds with $u' = u^{x_m}, v' = v^{x_m}$, and $\Delta = \bigcup_{j=1}^{m} \Delta^{x_j}$.

4.3. Exponent signatures of stacked letters

Consider the word $w$ in (4.1) with stacked letter $x$. The sequence $(e_1, \ldots, e_r)$ is said to be the exponent signature of $x$ in $w$ and is denoted by $\exp(x, w)$. It is convenient to define $\exp(y, w) = \emptyset$ whenever $y \not\in C(w)$. For any pair of stacked words $u$ and $v$, 


the expression \( \exp(u) = \exp(v) \) indicates the condition that \( \exp(x, u) = \exp(x, v) \) for all \( x \in \mathcal{X} \).

**Lemma 4.9.** Let \( u \approx v \) be any \( n \)-regular identity such that \( \Par(u) = \Par(v) \). Suppose that \( \exp(u) \neq \exp(v) \). Then

\[
Z_n\{u \approx v\} = Z_n\{\triangle, u' \approx v'\}
\]

(4.4)

for some set \( \triangle \) of identities from \( \mathbf{A}_n \) and some \( n \)-regular identity \( u' \approx v' \) such that \( \Par(u') = \Par(v') \) and \( \exp(u') = \exp(v') \).

**Proof.** Suppose that \( \exp(y, u) = \exp(y, v) \) if and only if \( y \notin \{x_1, \ldots, x_m\} \). Since \( u \approx v \) is \( n \)-regular, \( m(x_1, u) = m(x_1, v) = k \) for some \( k \) with \( 1 \leq k < n \). Further, by the assumption that \( \Par(u) = \Par(v) \), it follows that \( \sta(x_1, u) = \sta(x_1, v) = r \) for some \( r \geq 1 \). However, if \( r = 1 \), then \( \exp(x_1, u) = (k) = \exp(x_1, v) \) is a contradiction. Therefore \( r \geq 2 \), and \( u \) and \( v \) can be written as

\[
u = u_0 \prod_{i=1}^{r} (x_i^{e_i} u_i) \quad \text{and} \quad v = v_0 \prod_{i=1}^{r} (x_i^{e_i} v_i),
\]

where \( x_1^{e_1}, \ldots, x_r^{e_r} \) and \( x_1^{f_1}, \ldots, x_r^{f_r} \) are the \( x_1 \)-stacks of \( u \) and \( v \) respectively, and \( (e_1, \ldots, e_r) = \exp(x_1, u) \neq \exp(x_1, v) = (f_1, \ldots, f_r) \). For each \( i \in \{1, \ldots, r-1\} \), pick a letter \( h_i \) from \( u_i \) that is simple in \( u \) to form a partition sequence \( \bar{h} = (h_1, \ldots, h_{r-1}) \) of \( x_1 \) in \( u \). By assumption, \( h \) is also a partition sequence of \( x_1 \) in \( v \). Therefore the identity

\[
\tau_{x_1} : x_1^{e_1} \prod_{i=1}^{r-1} (h_i x_i^{e_i+1}) \approx x_1^{f_1} \prod_{i=1}^{r-1} (h_i x_i^{f_i+1})
\]

is deducible from \( u \approx v \) so that \( Z_n\{u \approx v\} = Z_n\{\tau_{x_1}, u \approx v\} \). Note that \( \tau_{x_1} \) is an identity from \( \mathbf{A}_n \). Let \( u^{x_1} = u \) and \( v^{x_1} = v_0 \prod_{i=1}^{r} (x_i^{e_i} v_i) \). Then it is easy to see that \( \tau_{x_1} \vdash v \approx v^{x_1} \), whence

\[
Z_n\{u \approx v\} = Z_n\{\tau_{x_1}, u^{x_1} \approx v^{x_1}\},
\]

where \( u^{x_1} \approx v^{x_1} \) is an \( n \)-regular identity such that \( \Par(u^{x_1}) = \Par(v^{x_1}) \) and \( \exp(y, u^{x_1}) = \exp(y, v^{x_1}) \) if and only if \( y \notin \{x_2, \ldots, x_m\} \).

Repeat the argument in the previous paragraph on the letter \( x_2 \) of the identity \( u^{x_1} \approx v^{x_1} \) to deduce that

\[
Z_n\{u^{x_1} \approx v^{x_1}\} = Z_n\{\tau_{x_2}, u^{x_2} \approx v^{x_2}\}
\]

for some identity \( \tau_{x_2} \) from \( \mathbf{A}_n \) and some \( n \)-regular identity \( u^{x_2} \approx v^{x_2} \) such that \( \Par(u^{x_2}) = \Par(v^{x_2}) \) and \( \exp(y, u^{x_2}) = \exp(y, v^{x_2}) \) if and only if \( y \notin \{x_3, \ldots, x_m\} \). It is easy to see how this procedure can be repeated on \( x_3, \ldots, x_m \) to obtain identities \( \tau_{x_3}, \ldots, \tau_{x_m} \) from \( \mathbf{A}_n \) and \( n \)-regular identities \( u^{x_3} \approx v^{x_3}, \ldots, u^{x_m} \approx v^{x_m} \) such that

\[
Z_n\{u^{x_1} \approx v^{x_1}\} = Z_n\{\tau_{x_1+1}, u^{x_1+1} \approx v^{x_1+1}\},
\]
Par(u^{x+1}) = Par(v^{x+1}), and \exp(y, u^{x+1}) = \exp(y, v^{x+1}) if and only if y does not belong to \{x_i+2, \ldots, x_m\}. Note that u^{x_m} \cong v^{x_m} is an n-regular identity such that Par(u^{x_m}) = Par(v^{x_m}) and \exp(u^{x_m}) = \exp(v^{x_m}). The equation in (4.4) now holds with u' = u^{x_m}, v' = v^{x_m}, and \Delta = \{x^{x_1}, \ldots, x^{x_m}\}.

4.4. n-orthodox identities

An n-regular identity u \cong v is said to be n-orthodox if it satisfies all of the following conditions:

(O1) Par(u) = Par(v) and \exp(u) = \exp(v) (so that sta(x, u) = sta(x, v) for all \( x \in \mathcal{X} \));

(O2) if sta(x, u) = sta(x, v) = 1 and y is any simple letter of both u and v, then x \prec u y if and only if x \prec v y.

Lemma 4.10. Suppose that u \cong v is any n-regular identity that satisfies condition (O1) but not condition (O2). Then

\[ Z_n\{u \cong v\} = Z_n\{\Sigma, u' \cong v'\} \tag{4.5} \]

for some subset \( \Sigma \) of \( \{x_k^n \mid 2 \leq k < n\} \) and some n-orthodox identity u' \cong v'.

Proof. Suppose that x_1, \ldots, x_m are all the letters of u and v such that for each i,

(a) sta(x_i, u) = sta(x_i, v) = 1;

(b) there exists some simple letter y_i of both u and v such that either x_i \prec u y_i or x_i \prec v y_i but not both.

Without loss of generality, assume that x_1 \prec u y_1 and x_1 \not\prec v y_1. It follows from condition (R2) that m(x_1, u) = m(x_1, v) = k_1 for some k_1 < n. If k_1 = 1, then x_1 is simple in u and v so that condition (R3) is contradictorily violated. Therefore 2 \leq k_1 < n. It follows that

\[ u = u_1 x_1^{k_1} u_2 y_1 u_3 \quad \text{and} \quad v = v_1 y_1 v_2 x_1^{k_1} v_3 \]

for some u_1, u_2, u_3, v_1, v_2, v_3 \in \mathcal{X}^* such that x_1, y_1 \notin C(u_i, v_i) for all i. The identity \( x_1^{k_1} \) is clearly deducible from u \cong v so that \( Z_n\{u \cong v\} = Z_n\{x_1^{k_1}, u \cong v\} \). The identity \( x_1^{k_1} \) can be used to move the x_1-stacks in u \cong v to the left resulting in the identity \( x_1^{k_1} u^{x_1} \cong x_1^{k_1} v^{x_1} \), where \( u^{x_1} = u_1 u_2 y_1 u_3 \) and \( v^{x_1} = v_1 y_1 v_2 v_3 \) are obtained from u and v, respectively, by removing all occurrences of x_1. Therefore

\[ Z_n\{u \cong v\} = Z_n\{x_1^{k_1}, x_1^{k_1} u^{x_1} \cong x_1^{k_1} v^{x_1}\} = Z_n\{x_2^{k_1}, u^{x_1} \cong v^{x_1}\} \]

where the last equality holds by Lemma 2.2.

Now u^{x_1} \cong v^{x_1} is not n-orthodox since sta(x_2, u^{x_1}) = sta(x_2, v^{x_1}) = 1 and y_2 is a simple letter of u^{x_1} and v^{x_1} such that either x_2 \prec u y_2 or x_2 \prec v y_2 but not both. Repeat the argument in the previous paragraph to obtain

\[ Z_n\{u^{x_1} \cong v^{x_1}\} = Z_n\{x_2^{k_1}, u^{x_2} \cong v^{x_2}\}, \]
where $k_2$ is some integer with $2 \leq k_2 < n$ and $u^{x_2} \approx v^{x_2}$ is obtained from $u^{x_1} \approx v^{x_1}$ by removing all occurrences of $x_2$. It is easy to see how this procedure can be repeated to obtain integers $k_3, \ldots, k_m$ and identities $u^{x_3} \approx v^{x_3}, \ldots, u^{x_m} \approx v^{x_m}$ such that
\[
Z_n(u^{x_i} \approx v^{x_i}) = Z_n(x_{k_i+1}^2, u^{x_{i+1}} \approx v^{x_{i+1}}),
\]
where $u^{x_{i+1}} \approx v^{x_{i+1}}$ is obtained from $u^{x_i} \approx v^{x_i}$ by removing all occurrences of $x_{i+1}$. Note that the identity $u^{x_m} \approx v^{x_m}$ is obtained from $u \approx v$ by removing all occurrences of the letters $x_1, \ldots, x_m$ and so is $n$-orthodox. The equation in (4.5) now holds with $u' = u^{x_m}$, $v' = v^{x_m}$, and $\Sigma = \{x_{k_1}^2, \ldots, x_{k_m}^2\}$. \hfill \Box

Define the relation $\sim$ on $X^*$ by $u \sim v$ if $m(x, u) = m(x, v)$ for all $x \in X$. Equivalently, $u \sim v$ if and only if $u$ and $v$ are identical up to ordering of their letters.

Lemma 4.11. Suppose that $u \approx v$ is any nontrivial $n$-orthodox identity. Then
\[
Z_n(u \approx v) = Z_n(\emptyset)
\]
for some set $\emptyset$ of identities from $\bullet_n$.

Some preparations are required to prove Lemma 4.11. Let $u \approx v$ be the $n$-orthodox identity in this lemma. If either $u$ or $v$ is simple, then the identity $u \approx v$ is contradictorily trivial by conditions (R2) and (R3). Therefore both $u$ and $v$ are non-simple. Since $u$ is a stacked word, it is of the form
\[
u = h_0 \prod_{i=1}^{m} (u_i, h_i)
\]
where $h_0, h_m \in X^*$, $h_1, \ldots, h_{m-1}, u_1, \ldots, u_m \in X^+$, and $m \geq 1$ are such that
(a) the letters of $h_0, \ldots, h_m \in X^*$ are precisely the simple letters of $u$;
(b) the letters of $u_1, \ldots, u_m$ are all non-simple in $u$;
(c) for each $i$, the word $u_i$ is a product of stacks of $u$.

(It follows from the definition of a stacked word that if $u_i = x_1^{e_1} \cdots x_r^{e_r}$ where $x_1^{e_1}, \ldots, x_r^{e_r}$ are stacks of $u$, then the letters $x_1, \ldots, x_r$ must be distinct.) Since $u \approx v$ is $n$-orthodox, $v$ is of the same form as $u$ above, that is,
\[
v = h_0 \prod_{i=1}^{m} (v_i, h_i)
\]
where $h_0, h_m \in X^*$, $h_1, \ldots, h_{m-1}, v_1, \ldots, v_m \in X^+$, and $m \geq 1$ are such that
(d) the letters of $h_0, \ldots, h_m \in X^*$ are precisely the simple letters of $v$;
(e) the letters of $v_1, \ldots, v_m$ are all non-simple in $v$;
(f) for each $i$, the word $v_i$ is a product of stacks of $v$.

Further, it follows from conditions (O1) and (O2) that for each $i$,
(g) \( u_i \sim v_i \), that is, \( x^e \) is an \( x \)-stack of \( u \) that occurs in \( u_i \) if and only if \( x^e \) is an \( x \)-stack of \( v \) that occurs in \( v_i \);

(h) the primary stacks of \( u \) that occur in \( u_i \) coincide with the primary stacks of \( v \) that occur in \( v_i \).

For the rest of this subsection, the words \( u \) and \( v \) will be rewritten on several occasions by invoking some identities of \( \mathbb{Z}_n \{ u \approx v \} \). To avoid excessive introduction of new variables, the words rewritten from \( u \) and \( v \) will often be denoted by \( u \) and \( v \) respectively without change in notation. This abuse of notation does not create any inaccuracy since the identity formed by the rewritten words will define, within \( \mathbb{Z}_n \), the same variety as \( \mathbb{Z}_n \{ u \approx v \} \).

It follows from (b) that for each \( i \), any secondary stack of \( u \) that appears in \( u_i \) can be moved by the identities (3.1) to any position within \( u_i \). Specifically, all secondary stacks of \( u \) that appear in \( u_i \) can be grouped to the left of \( u_i \) and arranged in alphabetical order. Hence \( u \) can be rewritten as

\[
\begin{align*}
\left(3.1\right) & \quad u = h_0 m \prod_{i=1}^{m} (s_i p_i h_i)
\end{align*}
\]

where the word \( p_i \) is a possibly empty product of some primary stacks of \( u \) that occur in \( u_i \), the word \( s_i \) is a possibly empty product of some secondary stacks of \( u \) that occur in \( u_i \), and the stacks in \( s_i \) are arranged in alphabetical order. It follows from (g) and (h) that all stacks of \( s_i \) are secondary stacks of \( v \) that occur in \( v_i \).

Therefore \( v \) can be rewritten as

\[
\begin{align*}
\left(3.1\right) & \quad v = h_0 m \prod_{i=1}^{m} (s_i q_i h_i)
\end{align*}
\]

where \( q_i \) is a possibly empty product of some primary stacks of \( v \) that occur in \( v_i \) (so that \( p_i \sim q_i \) by (g)). Note that \( s_1 = p_m = q_m = \emptyset \) necessarily. It is convenient to call \( p_i \) the \( i \)-th primary stack product of \( u \) and \( q_i \) the \( i \)-th primary stack product of \( v \).

**Lemma 4.12.** Suppose that \( \ell \) is any integer such that the \( \ell \)-th primary stack products of \( u \) and \( v \) are different, that is, \( p_\ell \neq q_\ell \). Then

\[
\mathbb{Z}_n \{ u \approx v \} = \mathbb{Z}_n \{ \diamond, u \approx v' \}
\]

for some set \( \diamond \) of identities from \( \mathbb{Z}_n \) and some word \( v' \) of the form (4.6) such that the \( \ell \)-th primary stack products of \( u \) and \( v' \) are identical.

**Proof.** Let \( z \in \mathcal{X}^\ast \) be the longest suffix that is common to both \( p_\ell \) and \( q_\ell \). Then \( p_\ell = ay/z \) and \( q_\ell = by/x_1^e \cdots x_r^e z \) for some \( a, b \in \mathcal{X}^\ast \) and distinct stacks \( x_1^e, \ldots, x_r^e, y^f \) with \( r \geq 1 \). It suffices to show how \( v \) can be rewritten, by invoking some identities of \( \mathbb{Z}_n \{ u \approx v \} \), into a word \( v' \) such that the \( \ell \)-th primary stack products of \( u \) and \( v' \) share the longer prefix \( y^f z \). This procedure can then be repeated to obtain the required word \( v' \).
For each $i$, since the stacks in $p_i$ and $q_i$ are primary in $u$ and $v$ respectively, it follows from (g) and (h) that $p_{i} \sim q_{i}$. Hence the stacks $x_{1}^{i}, \ldots, x_{r}^{i}$ of $q_{i}$ must appear in the factor $a$ of $p_i$. Further, the secondary $y$-stacks and secondary $x_1$-stacks of both $u$ and $v$ must occur in some of $s_{\ell+1}, \ldots, s_m$. Therefore, for $j > \ell$, if either an $x_1$-stack (say $x_{1}^{r}$) or a $y$-stack (say $y^{q}$) or both occur in $s_{j}$, then the identities (3.1) can be applied to group these stacks to the front of $s_{j}$ resulting in $w_{j}s_{j}'$, where $w_{j} \in \{x_{1}^{r}, y^{q}, x_{1}^{r}y^{q}\}$ and $s_{j}'$ is obtained from $s_{j}$ by removing all occurrences of $x_{1}$ and $y$.

Therefore

$$u^{(3.1)} = h_{0} \left( \prod_{i=1}^{\ell-1} (s_{i}, p_{i}, h_{i}) \right) s_{i} a y' f z h_{\ell} \prod_{i=\ell+1}^{m} \left( w_{i}s_{i}'(p_{i}, h_{i}) \right),$$

$$v^{(3.1)} = h_{0} \left( \prod_{i=1}^{\ell-1} (s_{i}, q_{i}, h_{i}) \right) s_{i} b y' x_{1}^{e_{1}} x_{2}^{e_{2}} \cdots x_{r}^{e_{r}} z h_{\ell} \prod_{i=\ell+1}^{m} \left( w_{i}s_{i}'(q_{i}, h_{i}) \right).$$

(4.7)

For each $i$, let $h_{i}$ be the first letter of $h_{i}$. Denote by $\alpha$ the substitution $z \mapsto 1$ for all $z \notin \{x_{1}, y, h_{\ell}, \ldots, h_{m-1}\}$. Then

$$u \approx v \vdash u \alpha \approx v \alpha$$

$$\vdash \lambda_{1} : x_{1}^{e_{1}} y^{f} \prod_{i=\ell}^{m-1} (h_{i}w_{i+1}) \approx y^{f} x_{1}^{e_{1}} \prod_{i=\ell}^{m-1} (h_{i}w_{i+1})$$

where $\lambda_{1}$ is some identity from $\delta_{n}$. Therefore $Z\{u \approx v\} = Z\{\lambda_{1}, u \approx v\}$. Now it is easy to see that the identity $\lambda_{1}$ of $Z\{u \approx v\}$ can be used to interchange the primary $y$-stack and primary $x_1$-stack of $v$ in (4.7) to obtain

$$v_{1} = h_{0} \left( \prod_{i=1}^{\ell-1} (s_{i}, q_{i}, h_{i}) \right) s_{i} b x_{1}^{e_{1}} y^{f} x_{2}^{e_{2}} \cdots x_{r}^{e_{r}} z h_{\ell} \prod_{i=\ell+1}^{m} \left( w_{i}s_{i}'(q_{i}, h_{i}) \right).$$

Therefore $Z\{u \approx v\} = Z\{\lambda_{1}, u \approx v_{1}\}$. If the factor $x_{2}^{e_{2}} \cdots x_{r}^{e_{r}}$ of $v_{1}$ is nonempty, then the procedure in this paragraph can be repeated to interchange the primary stacks $y^{f}$ and $x_{2}^{e_{2}}$ in $v_{1}$. Specifically, $Z\{u \approx v_{1}\} = Z\{\lambda_{2}, u \approx v_{2}\}$ for some identity $\lambda_{2}$ from $\delta_{n}$ and

$$v_{2} = h_{0} \left( \prod_{i=1}^{\ell-1} (s_{i}, q_{i}, h_{i}) \right) s_{i} b x_{1}^{e_{1}} x_{2}^{e_{2}} y^{f} x_{3}^{e_{3}} \cdots x_{r}^{e_{r}} z h_{\ell} \prod_{i=\ell+1}^{m} \left( w_{i}s_{i}'(q_{i}, h_{i}) \right).$$

Continuing in this manner, $Z\{u \approx v_{r-1}\} = Z\{\lambda_{r}, u \approx v_{r}\}$ for some identity $\lambda_{r}$ from $\delta_{n}$ and

$$v_{r} = h_{0} \left( \prod_{i=1}^{\ell-1} (s_{i}, q_{i}, h_{i}) \right) s_{i} b x_{1}^{e_{1}} \cdots x_{r}^{e_{r}} y^{f} z h_{\ell} \prod_{i=\ell+1}^{m} \left( w_{i}s_{i}'(q_{i}, h_{i}) \right).$$

Hence $Z\{u \approx v\} = Z\{\emptyset, u \approx v'\}$ where $v' = v_{r}$ and $\emptyset = \{\lambda_{1}, \ldots, \lambda_{r}\}$. □
Proof of Lemma 4.11. Since $\ell$ in Lemma 4.12 is arbitrary, the result can be repeated so that $Z_n\{u \approx v\} = Z_n\{\emptyset, u \approx v^\dagger\}$ for some set $\emptyset$ of identities from $\phi_n$ and some word $v^\dagger$ of the form (4.6) such that for any $\ell$, the $\ell$th-primary stack products of $u$ and $v^\dagger$ are identical. The identity $u \approx v^\dagger$ is then trivial so that $Z_n\{u \approx v\} = Z_n\{\emptyset\}$. \hfill \[\Box\]

4.5. Proof of Proposition 4.1

As noted earlier in this section, all subvarieties of $Z_1$ and $Z_2$ are finitely based by Lemma 2.5. Hence assume that $n \geq 3$ and let $V$ be any subvariety of $Z_n$. If $V$ is commutative, then it is already finitely based by Lemma 2.3. Therefore it suffices to assume that $V$ is noncommutative, whence by Corollary 4.5, there exist some set $\triangle$ of identities from $\triangle_n$ and some set $\mathcal{R}$ of $n$-regular identities such that $V = Z_n\{\triangle, \mathcal{R}\}$. It follows from Lemmas 4.8 and 4.9 that $Z_n\{\mathcal{R}\} = Z_n\{\triangle', \mathcal{R}'\}$ for some set $\triangle'$ of identities from $\triangle_n$ and some set $\mathcal{R}'$ of identities $u \approx v$ that satisfy condition (O1). Further, it follows from Lemmas 4.10 and 4.11 that $Z_n\{\mathcal{R}'\} = Z_n\{\Sigma, \emptyset\}$ for some subset $\Sigma$ of $\{s_k^n | 2 \leq k < n\}$ and some set $\emptyset$ of identities from $\phi_n$. Consequently, the variety

$$V = Z_n\{\triangle, \triangle', \Sigma, \emptyset\}$$

is finitely based.

Corollary 4.13. Each $Z_n$ contains only finitely many subvarieties.

Proof. By Lemma 2.5, the varieties $Z_1$ and $Z_2$ contain finitely many subvarieties. Therefore assume that $n \geq 3$. It follows from (4.8) that the number of noncommutative subvarieties of $Z_n$ is at most $2^m$, where $m$ is the total number of identities in $\{s_k^n | 2 \leq k < n\}$, and $\phi_n$. As for commutative subvarieties of $Z_n$, it is easy to show that they are $Z_k\{xy \approx yx\}$ where $k \in \{1, \ldots, n\}$. \hfill \[\Box\]

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References


