EDMOND W. H. LEE and JIAN RONG LI

Minimal non-finitely based monoids
Edmond W. H. Lee  
Department of Mathematics  
Simon Fraser University  
Burnaby, British Columbia V5A 1S6  
Canada  
Email: ewl@sfu.ca  

Jian Rong Li  
Department of Mathematics  
Lanzhou University  
Lanzhou, Gansu 730000  
People’s Republic of China  
Email: lijr@lzu.edu.cn
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Abstract

Two semigroups are said to be distinct if they are neither isomorphic nor anti-isomorphic. Although there exist 1373 distinct monoids of order six, only two are known to be non-finitely based. In the present dissertation, the finite basis property of the other 1371 distinct monoids of order six is verified. Since it is long established that all semigroups of order five or less are finitely based, the two known non-finitely based monoids of order six are the only examples of minimal order.

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1. Introduction

1.1. The finite basis problem for small semigroups. A semigroup is \textit{finitely based} if there exists a finite set of its identities from which all its other identities can be deduced. In 1969, Perkins \cite{Perkins1969} published the first two examples of non-finitely based finite semigroups, the first of which is the monoid \(B_2\) of order six obtained by adjoining an identity element to the Brandt semigroup

\[ B_2 = \langle a, b \mid a^2 = b^2 = 0, aba = a, bab = b \rangle \]

of order five. The discovery of a non-finitely based semigroup with only six elements focused much attention upon the finite basis problem for semigroups of order five or less. This problem was explicitly raised by Tarski \cite{Tarski1966} in 1966 and attracted the interest of Bol’bot \cite{Bolbot1973}, Edmunds \cite{Edmunds1974, Edmunds1975}, Karnofsky \cite{Karnofsky1978}, Tishchenko \cite{Tishchenko1980}, and Trahtman \cite{Trahtman1981}. A solution to this problem was eventually completed by Trahtman \cite{Trahtman1991, Trahtman1992} in the early 1980s and published in 1991 \cite{Trahtman1991}.

\textbf{Theorem 1.1.} Every semigroup of order five or less is finitely based.

A more complete historical account of the proof of Theorem \textbf{1.1} can be found in the survey of Shevrin and Volkov \cite{Shevrin1995, §10}.

By the late 1980s, two more non-finitely based semigroups of order six were discovered: \(A_2\) and \(A_2^g\). The semigroup \(A_2\), due to Sapir \cite{Sapir1980} and Trahtman \cite{Trahtman1983} independently, is the monoid obtained by adjoining an identity element to the 0-simple semigroup

\[ A_2 = \langle a, b \mid a^2 = aba = a, b^2 = 0, bab = b \rangle \]

of order five. The semigroup \(A_2^g\), due to Volkov \cite{Volkov1985}, is obtained by adjoining a new element \(g\) to the semigroup \(A_2\) with multiplication defined by \(g^2 = 0\) and \(gx = xg = g\) for all \(x \in A_2\).

Two semigroups are said to be \textit{distinct} if they are neither isomorphic nor anti-isomorphic. It follows from Theorem \textbf{1.1} that the semigroups \(A_2^g, A_2^1,\) and \(B_2^1\) of order six are minimal with respect to being non-finitely based. In fact, up to the present, these three non-finitely based semigroups are the only known minimal examples. Since there exist 15973 distinct semigroups of order six \cite{Shevrin1995} among which 1373 are monoids \cite{Edmunds1974}, it is very natural to question the existence of non-finitely based semigroups or monoids of order six that are distinct from \(A_2^g, A_2^1,\) and \(B_2^1\). The objective of the present dissertation is to provide an answer to this question for the case of monoids.

\textsuperscript{(1)} The non-finite basis property of the semigroup \(A_2^g\) also follows from Mashevitskiï \cite{Mashevitski1978}. Refer to Lee and Volkov \cite{Lee1993} for an easily described basis of the semigroup \(A_2^g\).
Main Theorem. Every monoid of order six distinct from $A_2^1$ and $B_2^1$ is finitely based. Consequently, up to isomorphism and anti-isomorphism, $A_2^1$ and $B_2^1$ are the only non-finitely based monoids of minimal order.

The ultimate goal of completely identifying all minimal non-finitely based semigroups clearly requires the solution of the finite basis problem for all distinct nonunital semigroups of order six. This is a nontrivial and potentially daunting task since the number of such semigroups is $15973 - 1373 = 14600$.

1.2. Organization. The set of all chapters in the present dissertation under the prerequisite relation constitutes the directed tree in Figure 1. Every chapter that follows Chapter 2 can be read independently.

![Fig. 1. Prerequisites of chapters](image)

Notation and background information are given in Chapter 2. An outline of the proof of the main theorem is given in Chapter 3 while the finer details are deferred to Chapters 4–15. For completeness, multiplication tables of all 1373 distinct monoids of order six are listed in Appendix A. Each finitely based monoid is associated with a result from Chapter 3 that guarantees its finite basis property.

2. Preliminaries

2.1. Letters and words. Let $\mathcal{X}$ be a countably infinite alphabet. For any subset $\mathcal{Y}$ of $\mathcal{X}$, let $\mathcal{Y}^+$ and $\mathcal{Y}^*$ denote the free semigroup and free monoid over $\mathcal{Y}$ respectively. Elements of $\mathcal{X}$ are referred to as letters and elements of $\mathcal{X}^+$ and $\mathcal{X}^*$ are referred to as words. Let $x$ be any letter and $w$ be any word. Then

- the leftmost letter of $w$ is denoted by $\lambda(w)$;
- the content of $w$, denoted by $\text{con}(w)$, is the set of letters occurring in $w$;
- the number of occurrences of $x$ in $w$ is denoted by $\text{occ}(x, w)$;
- $x$ is simple in $w$ if $\text{occ}(x, w) = 1$;
- the set of simple letters of $w$ is denoted by $\text{sim}(w)$;
- $w$ is simple if each of its letters is simple in it, that is, $\text{sim}(w) = \text{con}(w)$;
- the initial of $w$, denoted by $\text{ini}(w)$, is the simple word obtained from $w$ by retaining the first occurrence of each letter.

Define a relation $\cong$ on $\mathcal{X}^+$ by $u \cong v$ if $\text{occ}(x, u) = \text{occ}(x, v)$ for all $x \in \mathcal{X}$. Equivalently, $u \cong v$ if and only if $u$ and $v$ are the same word up to letter rearrangement.
2.2. Identities for some semigroups. An identity $u \approx v$ is nontrivial if $u$ and $v$ are distinct words. For any identity $u \approx v$, denote by $u^* \approx v^*$ the system of all nontrivial identities that can be obtained from $u \approx v$ by removing all occurrences of some letters. For example, $xyxzx^* \approx xyxz$ denotes the system
\[
\{ xyxzx \approx xyxz, xyx^2 \approx xyx, x^2zx \approx x^2z, x^3 \approx x^2 \}.
\]
A semigroup $S$ satisfies an identity $u \approx v$ if $u \phi = v \phi$ for any substitution $\phi$ from $X$ into $S$. For any set $\Sigma$ of identities, the variety defined by $\Sigma$ is the class of all semigroups that satisfy all identities in $\Sigma$; in this case, $\Sigma$ is a basis for the variety. A variety is finitely based if it possesses a finite basis. All varieties in the present dissertation are varieties of semigroups. Refer to Burris and Sankappanavar [5], Shevrin and Volkov [34], and Volkov [44] for any concept of universal algebra that appears here but is undefined.

For any semigroup $S$, let $S^1$ be the monoid obtained from $S$ by adjoining an identity element.

**Lemma 2.1** ([2, Lemma 7.1.1]). Let $M$ be any variety generated by a monoid. Suppose that $S$ is any nonunital semigroup in the variety $M$. Then the monoid $S^1$ belongs to the variety $M$.

The left-zero semigroup $L^2$ of order two, the null semigroup $N_2$ of order two, and the cyclic group $Z_n$ of order $n$ can be given by the following presentations:
\[
L_2 = \langle a, b \mid a^2 = ab = a, b^2 = ba = b \rangle,
\]
\[
N_2 = \langle a \mid a^2 = 0 \rangle,
\]
\[
Z_n = \langle a \mid a^n = 1 \rangle.
\]

**Lemma 2.2.** Let $u \approx v$ be any identity. Then

(i) $L^2_1$ satisfies $u \approx v$ if and only if $\text{ini}(u) = \text{ini}(v)$;
(ii) $N^2_1$ satisfies $u \approx v$ if and only if $\text{con}(u) = \text{con}(v)$ and $\text{sim}(u) = \text{sim}(v)$;
(iii) $Z_n$ satisfies $u \approx v$ if and only if $\text{occ}(x, u) \equiv \text{occ}(x, v) \pmod{n}$ for all $x \in X$.

**Proof.** These results are well known and easy to verify. $\blacksquare$

**Lemma 2.3.** Let $M$ be any noncommutative monoid such that the monoid $N^2_1$ is embeddable in $M$ and let $u \approx v$ be any identity satisfied by $M$. If either $u$ or $v$ is a simple word, then the identity $u \approx v$ is trivial.

**Proof.** By symmetry, it suffices to assume that $u$ is a simple word. By assumption, the identity $u \approx v$ is also satisfied by the monoid $N^2_1$. Therefore the word $v$ is simple by Lemma 2.2(ii). Since the monoid $M$ is noncommutative, it is easy to see that the words $u$ and $v$ are identical. $\blacksquare$

2.3. Exclusion identities. Let $S$ be any semigroup in a variety $V$. Suppose that an identity $u \approx v$ is satisfied by any subvariety of $V$ that does not contain the semigroup $S$. Then $u \approx v$ is an exclusion identity for $S$ in $V$.

**Lemma 2.4.** Let $S$ be any semigroup in a periodic variety $V$ such that the monoid $S^1$ also belongs to $V$. Suppose that $u \approx v$ is an exclusion identity for $S$ in $V$. Then any
subvariety of \( V \) generated by a monoid that does not contain the monoid \( S^1 \) must satisfy the identity \( u \approx v \).

**Proof.** By assumption, the variety \( V \) satisfies the identity \( x^{2n} \approx x^n \) for some \( n \geq 1 \). Choose any \( h \in \mathcal{X} \setminus \text{con}(uv) \). Denote by \( \psi \) the substitution \( x \mapsto h^n x h^n \) for all \( x \in \text{con}(uv) \). Since \( u \approx v \) is an exclusion identity for \( S \) in \( V \), it follows from Lee [18, Theorem 2] that \( u\psi \approx v\psi \) is an exclusion identity for \( S^1 \) in \( V \). Therefore any subvariety \( M \) of \( V \) generated by a monoid that does not contain the monoid \( S^1 \) must satisfy the identity \( u\psi \approx v\psi \); it is easy to see that the variety \( M \) also satisfies the identity \( u \approx v \). ■

2.4. **Precedence.** Let \( x \) and \( y \) be any letters of a word \( w \). Then

- the number of \( x \) in \( w \) that precedes the first \( y \) in \( w \) is denoted by \( \text{occ}(x,y,w) \);
- write \( x \ll_w y \) if every \( x \) in \( w \) precedes every \( y \) in \( w \).

An identity \( u \approx v \) is said to preserve complete precedence if \( \text{con}(u) = \text{con}(v) \) and the following condition holds: \( x \ll_u y \) if and only if \( x \ll_v y \) for any \( x,y \in \mathcal{X} \).

**Lemma 2.5** ([8, Lemma 4.2]). Let \( \text{pur} \) and \( \text{pu}'r \) be any words such that

(a) \( u \triangleleft u' \);
(b) \( \text{occ}(x, \text{pur}) \geq 2 \) for all \( x \in \text{con}(u) \);
(c) \( \text{pur} \approx \text{pu}'r \) preserves complete precedence.

Then the identity system \( \{xhytxy \ast \approx xhytyx, xyhxty \ast \approx yxhxty\} \) implies the identity \( \text{pur} \approx \text{pu}'r \).

3. **Proof of the main theorem**

The following nine sufficient conditions for the finite basis property are required:

**Condition 1** (Pollák [30, Theorem 1]). Any semigroup that satisfies the identity

\[ xyx \approx x^2 y \]

is finitely based.

**Condition 2** (Rasin [31]). Any finite orthodox completely regular semigroup is finitely based. Specifically, any finite semigroup that satisfies the identities

\[ x^{13} \approx x, \quad x^{12} y^{12} x^{12} y^{12} \approx x^{12} y^{12} \]

is finitely based.

**Condition 3** (Edmunds et al. [10, Theorem 7.2]). Any monoid that satisfies the identities

\[ x^2 y \approx y x^2, \quad x^3 y x \approx x y x, \quad x y x y \approx x^2 y^2 \]

is finitely based.

**Condition 4.** Any monoid that satisfies the identities

\[ x^2 y x \approx x y x, \quad x y x^2 \approx x y x, \quad x^2 y^2 \approx y^2 x^2 \]

is finitely based.
CONDITION 5 (Lee [21, Corollary 3.4]). Any monoid that satisfies the identities
\[ xyxy \approx xy^2x, \ xyxxz \approx xyzx \]
is finitely based.

CONDITION 6. Any monoid that satisfies the identities
\[ x^3yx \approx xyx, \ xy^2x \approx yx^2y, \ xyxxz \approx x^2yzx \]
is finitely based.

CONDITION 7 (Luo and Zhang [25, Theorem 1.1 and Corollary 4.6]). Any monoid that satisfies the identities
\[ x^7yx \approx xyx, \ xyxy \approx x^2y^2x, \ xyxxz \approx x^2yzx \]
is finitely based.

CONDITION 8. Any monoid that satisfies the identities
\[ xyx^3 \approx xyx, \ xyxxz \approx xyx^2z, \ xhxy \approx xhxy \]
but violates the identity
\[ xyxy \approx x^2y^2 \]
is finitely based.

CONDITION 9. Any semigroup that satisfies the identities
\[ x^4 \approx x^2, \ x^3yx \approx xyx, \ xyx^2 \approx x^3y, \ xyxy \approx x^2y^2 \]
but violates both of the identities
\[ xyxy \approx x^2y^2, \ xyx^3 \approx xyx \]
is finitely based.

Conditions 4, 6, 8, and 9 are established in Chapters 4–7.

With the aid of a computer, it is routine to show that 1360 of the 1373 distinct monoids of order six are finitely based by Conditions 1–9 or their dual conditions. The remaining 13 sporadic cases are the monoids \( A, B, \ldots, K, B_1, \) and \( A_1 \) given by the following multiplication tables:

\[
\begin{array}{cccccc}
A & 1 & 2 & 3 & 4 & 5 & 6 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 & 1 & 1 & 1 \\
3 & 1 & 1 & 1 & 1 & 1 & 1 \\
4 & 1 & 1 & 1 & 1 & 1 & 1 \\
5 & 1 & 1 & 1 & 1 & 1 & 1 \\
6 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{cccccc}
B & 1 & 2 & 3 & 4 & 5 & 6 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 & 1 & 1 & 1 \\
3 & 1 & 1 & 1 & 1 & 1 & 1 \\
4 & 1 & 1 & 1 & 1 & 1 & 1 \\
5 & 1 & 1 & 1 & 1 & 1 & 1 \\
6 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{cccccc}
C & 1 & 2 & 3 & 4 & 5 & 6 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 & 1 & 1 & 1 \\
3 & 1 & 1 & 1 & 1 & 1 & 1 \\
4 & 1 & 1 & 1 & 1 & 1 & 1 \\
5 & 1 & 1 & 1 & 1 & 1 & 1 \\
6 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{cccccc}
D & 1 & 2 & 3 & 4 & 5 & 6 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 & 1 & 1 & 1 \\
3 & 1 & 1 & 1 & 1 & 1 & 1 \\
4 & 1 & 1 & 1 & 1 & 1 & 1 \\
5 & 1 & 1 & 1 & 1 & 1 & 1 \\
6 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{cccccc}
E & 1 & 2 & 3 & 4 & 5 & 6 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 & 1 & 1 & 1 \\
3 & 1 & 1 & 1 & 1 & 1 & 1 \\
4 & 1 & 1 & 1 & 1 & 1 & 1 \\
5 & 1 & 1 & 1 & 1 & 1 & 1 \\
6 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{cccccc}
F & 1 & 2 & 3 & 4 & 5 & 6 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 & 1 & 1 & 1 \\
3 & 1 & 1 & 1 & 1 & 1 & 1 \\
4 & 1 & 1 & 1 & 1 & 1 & 1 \\
5 & 1 & 1 & 1 & 1 & 1 & 1 \\
6 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{cccccc}
G & 1 & 2 & 3 & 4 & 5 & 6 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 & 1 & 1 & 1 \\
3 & 1 & 1 & 1 & 1 & 1 & 1 \\
4 & 1 & 1 & 1 & 1 & 1 & 1 \\
5 & 1 & 1 & 1 & 1 & 1 & 1 \\
6 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{cccccc}
H & 1 & 2 & 3 & 4 & 5 & 6 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 & 1 & 1 & 1 \\
3 & 1 & 1 & 1 & 1 & 1 & 1 \\
4 & 1 & 1 & 1 & 1 & 1 & 1 \\
5 & 1 & 1 & 1 & 1 & 1 & 1 \\
6 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{cccccc}
I & 1 & 2 & 3 & 4 & 5 & 6 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 & 1 & 1 & 1 \\
3 & 1 & 1 & 1 & 1 & 1 & 1 \\
4 & 1 & 1 & 1 & 1 & 1 & 1 \\
5 & 1 & 1 & 1 & 1 & 1 & 1 \\
6 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{cccccc}
J & 1 & 2 & 3 & 4 & 5 & 6 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 & 1 & 1 & 1 \\
3 & 1 & 1 & 1 & 1 & 1 & 1 \\
4 & 1 & 1 & 1 & 1 & 1 & 1 \\
5 & 1 & 1 & 1 & 1 & 1 & 1 \\
6 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{cccccc}
K & 1 & 2 & 3 & 4 & 5 & 6 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 & 1 & 1 & 1 \\
3 & 1 & 1 & 1 & 1 & 1 & 1 \\
4 & 1 & 1 & 1 & 1 & 1 & 1 \\
5 & 1 & 1 & 1 & 1 & 1 & 1 \\
6 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{cccccc}
B_1 & 1 & 2 & 3 & 4 & 5 & 6 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 & 1 & 1 & 1 \\
3 & 1 & 1 & 1 & 1 & 1 & 1 \\
4 & 1 & 1 & 1 & 1 & 1 & 1 \\
5 & 1 & 1 & 1 & 1 & 1 & 1 \\
6 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{cccccc}
A_1 & 1 & 2 & 3 & 4 & 5 & 6 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 & 1 & 1 & 1 \\
3 & 1 & 1 & 1 & 1 & 1 & 1 \\
4 & 1 & 1 & 1 & 1 & 1 & 1 \\
5 & 1 & 1 & 1 & 1 & 1 & 1 \\
6 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]
As mentioned in the introduction, the monoids $A_2^1$ and $B_2^1$ are non-finitely based. The monoid $C$ is known to be finitely based [10, Theorem 3.2]. The finite basis property of the other ten monoids is verified in Chapters 8–15. The proof of the main theorem is thus complete.

Multiplication tables of all 1373 distinct monoids of order six are listed in Appendix A. The 13 sporadic cases above, together with other monoids satisfying certain special properties, are explicitly identified.

Remark 3.1.

(i) The finite basis property of the monoids $A$ and $E$ was first announced by Edmunds et al. [10, Section 3].

(ii) Conditions 1–9 are all required in the proof of the main theorem since they are distinguished by the monoids $M_1,\ldots,M_9$ with multiplication tables given below, that is, the monoid $M_m$ satisfies Condition $n$ if and only if $m = n$:

<table>
<thead>
<tr>
<th>$M_1$</th>
<th>$M_2$</th>
<th>$M_3$</th>
<th>$M_4$</th>
<th>$M_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$M_6$</th>
<th>$M_7$</th>
<th>$M_8$</th>
<th>$M_9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>1</td>
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</tr>
<tr>
<td>5</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

4. On Condition 4

The main aim of the present chapter is to establish the finite basis property of any monoid that satisfies the identities

$$x^2yx \approx xyx, \quad xyx^2 \approx xyx, \quad x^2y^2 \approx y^2x^2.$$  \(4.1\)

The semigroup

$$Q = \langle a, b, c \mid a^2 = a, \ ab = b, \ ca = c, \ ac = ba = cb = 0 \rangle$$

of order five plays a central role. Denote by $Q^1$ the variety generated by the monoid $Q^1$.

Remark 4.1. The semigroup $Q$ has appeared in Almeida’s investigation of minimal nonpermutative pseudovarieties [2, Chapter 6].

4.1. A basis for the variety $Q^1$

Lemma 4.2 ([2, Proposition 6.5.10]). The following statements on any identity $u \approx v$ satisfied by the semigroup $Q$ are equivalent:

(a) $xy$ is a factor of the word $u$ with $x, y \in \text{sim}(u)$;
(b) $xy$ is a factor of the word $v$ with $x, y \in \text{sim}(v)$. 

Proposition 4.3. The identities
\[ x^3 \approx x^2, \quad x^2yx \approx xyx, \quad xyx^2 \approx xyx, \quad x^2y^2 \approx y^2x^2 \] (4.2a)
constitute a basis for the variety \(Q^1\).

Proof. It is routine to verify that the variety \(Q^1\) satisfies the identities (4.2). It remains
show that any identity \(w \approx w'\) satisfied by the variety \(Q^1\) is implied by the identities (4.2).
Since the monoid \(Q^1\) is noncommutative and its submonoid \(\{0, b, 1\}\) is isomorphic to the
monoid \(N^1_2\), it follows from Lemma 2.2(ii) that the identity \(w \approx w'\) is trivial if either \(w\)
or \(w'\) is a simple word. Therefore assume that the words \(w\) and \(w'\) are both nonsimple, whence it is possible to write
\[ w = \prod_{i=1}^{m}(s_iw_i) \quad \text{and} \quad w' = \prod_{i=1}^{m'}(s'_iw'_i) \]
where
- the letters of \(s_i \in X^*\) and \(s_2, \ldots, s_m \in X^+\) are all simple in \(w\);
- the letters of \(w_1, \ldots, w_{m-1} \in X^+\) and \(w_m \in X^*\) are all nonsimple in \(w\);
- the letters of \(s'_i \in X^*\) and \(s'_2, \ldots, s'_{m'} \in X^+\) are all simple in \(w'\);
- the letters of \(w'_1, \ldots, w'_{m'-1} \in X^+\) and \(w'_{m'} \in X^*\) are all nonsimple in \(w'\).

Since the submonoid \(\{0, b, 1\}\) of \(Q^1\) is isomorphic to the monoid \(N^1_2\), it follows from
Lemma 2.2(ii) that \(\bigcup_{i=1}^{m}\, \mathrm{con}(s_i) = \bigcup_{i=1}^{m'}\, \mathrm{con}(s'_i)\) and \(\bigcup_{i=1}^{m}\, \mathrm{con}(w_i) = \bigcup_{i=1}^{m'}\, \mathrm{con}(w'_i)\). Further,
if \(m = m'\) and \(s_i = s'_i\) for all \(i\), whence
\[ w' = \prod_{i=1}^{m}(s_iw'_i). \]
If \(m = 1\), then clearly \(\mathrm{con}(w_1) = \mathrm{con}(w'_1)\). Therefore assume that \(m \geq 2\).

Case 1: \(\mathrm{con}(w_1) \neq \mathrm{con}(w'_1)\). Without loss of generality, assume that \(x \in \mathrm{con}(w_1)\) and
\(x \not\in \mathrm{con}(w'_1)\). Let \(z = \lambda(s_2)\) and let \(\varphi_1\) be the substitution \(h \mapsto 1\) for all \(h \in X \setminus \{x, z\}\).
Then the monoid \(Q^1\) satisfies the identity \(xy(w(x)\varphi_1) \approx xy(w'(x)\varphi_1)\), which is of the form
\(xyzx^q \approx xyzx^r\) for some \(p, q \geq 1\) and \(r \geq 2\).

Case 2: \(\mathrm{con}(w_m) \neq \mathrm{con}(w'_m)\). By an argument symmetrical to Case 1, the monoid \(Q^1\)
satisfies an identity of the form \(x^p yx^q z x^r \approx x^r y z x^p\) for some \(p, q \geq 1\) and \(r \geq 2\).

Case 3: \(\mathrm{con}(w_i) \neq \mathrm{con}(w'_i)\) for some \(i\) such that \(1 < i < m\). Without loss of generality,
avsume that \(x \in \mathrm{con}(w_i)\) and \(x \not\in \mathrm{con}(w'_i)\). Let \(y = \lambda(s_i)\) and \(z = \lambda(s_{i+1})\), and let \(\varphi_2\) be
the substitution \(h \mapsto 1\) for all \(h \in X \setminus \{x, y, z\}\). Then the monoid \(Q^1\) satisfies the identity
\((xw(x)\varphi_2 \approx (xw'(x)\varphi_2)\), which is of the form \(x^pyx^q z x^r \approx x^r y z x^p\) for some \(p, q, r, s, t \geq 1\).

The three cases just considered are all impossible since by Lemma 4.2, the monoid \(Q^1\)
cannot satisfy any identity of the form \(x^p yx^q z x^r \approx x^r y z x^p\). Therefore
\(\mathrm{con}(w_i) = \mathrm{con}(w'_i)\) for every \(i \in \{1, \ldots, m\}\). Let \(\varphi_3\) be the substitution \(x \mapsto x^2\) for all \(x \in \bigcup_{i=1}^{m}\, \mathrm{con}(w_i) = \bigcup_{i=1}^{m}\, \mathrm{con}(w'_i)\). Then the identities (4.2a)
imply the identities \(w\varphi_3 \approx w\) and \(w'\varphi_3 \approx w'\).
Further, for each \(i\), since \(w_i \varphi_3\) and \(w'_i \varphi_3\) are products of squares with \(\text{con}(w_i) = \text{con}(w'_i)\), it is easy to see that the identities (4.2) imply the identity \(w_i \varphi_3 \approx w'_i \varphi_3\). Consequently, \(w \approx w'\), that is, the identities (4.2) imply the identity \(w \approx w'\).

4.2. An exclusion identity

**Lemma 4.4.** The identity

\[
xyxz \approx xyzx
\]

(4.3)

is an exclusion identity for the semigroup \(Q\) in the variety \(Q^1\).

**Proof.** Let \(V\) be any subvariety of \(Q^1\) such that \(Q / V \not\in V\). Then it follows from Almeida [2, Lemma 6.5.14] that the variety \(V\) satisfies either the identity (4.3) or the identity

\[
xyxyx \approx xyx
\]

(4.4) If \(V\) satisfies (4.4), then it also satisfies (4.3) since

\[
xyxz \approx xyxz \approx xy^{x^2}x^2xy^2x^2xz \approx xyz^2x^2y^2zx \approx xyzxyxz \approx xyzxz
\]

Therefore (4.3) is an exclusion identity for the semigroup \(Q\) in the variety \(Q^1\).

4.3. Proof of Condition 4

Let \(M\) be any monoid that satisfies the identities (4.1) and let \(M\) be the variety generated by \(M\). It is clear from Proposition 4.3 that \(M\) is a subvariety of \(Q^1\). If \(Q^1 \in M\), then \(M = Q^1\) and \(M\) is finitely based by Proposition 4.3. Therefore assume that \(Q^1 \not\in M\). Then it follows from Lemmas 2.4 and 4.4 that the monoid \(M\) satisfies the identity (4.3). It also satisfies the identity \(xyxy \approx xy^2x\) since

\[
xyxy \approx xyxy^2 \approx xy^3x^2 \approx xy^2x
\]

Therefore \(M\) satisfies the identities in Condition 5 and is finitely based.

5. On Condition 6

It is easily seen that Condition 6 is a special case of the following result.

**Proposition 5.1.** Any monoid that satisfies the identities

\[
x^7yx \approx xyx, \quad xy^2x \approx yx^2y, \quad xyxz \approx x^2yzx
\]

(5.1)

is finitely based.

The proof of Proposition 5.1 is given in Section 5.3. The semigroup

\[A_0 = \langle a, b | a^2 = a, b^2 = b, ba = 0 \rangle\]

of order four plays a central role in this proof. Denote by \(A_0^1\) the variety generated by the monoid \(A_0^1\).
**Remark 5.2.**


(ii) The semigroup $A_0$ is not completely $0$-simple, but it is very important to the study of varieties generated by completely $0$-simple semigroups [15, 16, 19, 22, 23, 32].

(iii) The monoid $A_0^1$ has appeared in Edmunds [8] as $M_{20}$, while the subvarieties of $A_0^1$ were extensively investigated by Lee [20].

### 5.1. A basis for the variety $A_0^1 \lor \mathbb{Z}_n$

**Proposition 5.3.** Let $n \geq 2$ be any integer. Then the identities

\[
xyxzx \simeq x^2yzx, \quad (5.2a)
\]
\[
x^{n+1}yx \simeq x^nyx, \quad (5.2b)
\]
\[
xyttxy \simeq xyttx, \quad xyxtxy \simeq xhtxy
\]

constitute a basis for the variety $A_0^1 \lor \mathbb{Z}_n$.

In this chapter, a nonsimple word

\[
w = \prod_{i=1}^{m} (s_i w_i)
\]

is said to be in *n-canonical form* if all of the following conditions hold:

(I) the letters of $s_1 \in X^*$ and $s_2, \ldots, s_m \in X^+$ are all simple in $w$;

(II) the letters of $w_1, \ldots, w_{m-1} \in X^+$ and $w_m \in X^*$ are all nonsimple in $w$;

(III) $\text{occ}(x, w) \leq n + 1$ for all $x \in X$;

(IV) if $\text{occ}(x, w) = r + 1$ for some $r \in \{1, \ldots, n\}$, then either $\text{occ}(x, w_i) = r + 1$ for some $i$, or $\text{occ}(x, w_j) = r$ and $\text{occ}(x, w_k) = 1$ for some $j$ and $k$ with $j < k$.

Note that (I) and (II) imply that $\text{con}(s_i) \cap \text{con}(w_j) = \emptyset$ for any $i$ and $j$.

**Lemma 5.4.** Let $w$ be any nonsimple word. Then there exists some word $w'$ in *n-canonical form* such that the identities (5.2a) and (5.2b) imply the identity $w \simeq w'$.

**Proof.** It suffices to convert the nonsimple word $w$, using the identities (5.2a) and (5.2b), into a word in *n-canonical form*. By gathering adjacent simple letters and adjacent non-simple letters in the word $w$, it is easy to see that $w$ can be written in the form (5.3) that satisfies (I) and (II). Suppose that $\text{occ}(x, w) = e + 1$. Then any $x$ in the word $w$, except the first and last occurrences of $x$, can be gathered by the identities (5.2a) with the first occurrence of $x$, resulting in a word of the form $px^{e-1}qxr$, where $p, q, r \in (X \setminus \{x\})^*$. If necessary, the identities (5.2b) can then be used to reduce the exponent $e - 1$ to a number in $\{1, \ldots, n\}$. Repeat the same procedure on any letter that occurs more than twice in the word $w$. The resulting word then satisfies both (III) and (IV).

**Lemma 5.5 (8, Lemma 4.1).** Any identity satisfied by the monoid $A_0^1$ preserves complete precedence.
Proof of Proposition 5.3. It is routine to verify that the variety $A_0^1 \lor Z_n$ satisfies the identities (5.2). It remains to show that any identity $w \approx w'$ satisfied by the variety $A_0^1 \lor Z_n$ is implied by the identities (5.2). Since the monoid $A_0^1$ is noncommutative and its submonoid $\{0, ab, 1\}$ is isomorphic to the monoid $N_2$, it follows from Lemma 2.3 that the identity $w \approx w'$ is trivial if either $w$ or $w'$ is a simple word. Hence assume that the words $w$ and $w'$ are both nonsimple. In view of Lemma 5.4, these words can be assumed to be in $n$-canonical form, say

$$w = \prod_{i=1}^m (s_i w_i) \quad \text{and} \quad w' = \prod_{i=1}^{m'} (s'_i w'_i).$$

It follows from (III) and Lemma 2.2(ii) that $\text{occ}(x, w) = \text{occ}(x, w') \leq n + 1$ for all $x \in X$. Further, since the identity $w \approx w'$ preserves complete precedence by Lemma 5.5, it follows that $m = m'$, $s_i = s'_i$, and $w_i = w'_i$ for every $i$, whence

$$w' = \prod_{i=1}^m (s_i w'_i).$$

If $w_i = w'_i$ for all $i$, then the identity $w \approx w'$ is trivial and so is satisfied by the variety $A_0^1 \lor Z_n$. Suppose that $\ell$ is the least integer such that $w_\ell \neq w'_\ell$. Let

$$p = \left( \prod_{i=1}^{\ell-1} (s_i w_i) \right) s_\ell, \quad r = \prod_{i=\ell+1}^m (s_i w_i), \quad \text{and} \quad r' = \prod_{i=\ell+1}^{m'} (s_i w'_i).$$

Then $w = p w_\ell r$ and $w' = p w'_\ell r'$ with $r \neq r'$. It is routine to show that the identity $p w_\ell r \approx p w'_\ell r'$ preserves complete precedence. Hence by Lemma 2.5, the identities (5.2c) imply the identity $p w_\ell r \approx p w'_\ell r'$, that is,

$$w = s_1 w_1 \cdots s_\ell w_\ell \cdots s_m w_m \approx s_1 w_1 \cdots s_\ell w'_\ell \cdots s_m w_m.$$

It is easy to see how the same argument can be repeated on $w_{\ell+1}, \ldots, w_m$ to obtain

$$w \approx s_1 w_1 \cdots s_{\ell-1} w_{\ell-1} s_\ell w'_\ell s_{\ell+1} w'_{\ell+1} \cdots s_m w_m = w'.$$

Consequently, the identities (5.2) imply the identity $w \approx w'$. ■

5.2. Subvarieties of the variety $A_0^1 \lor Z_6$

Lemma 5.6. Let $n \geq 2$ be any integer. Then the identity

$$x^n y^n x^n y^n \approx x^n y^n$$

is an exclusion identity for the semigroup $A_0$ in the variety $A_0^1 \lor Z_n$.

Proof. This follows from Volkov [43, Proposition 1.2]. ■

Lemma 5.7. Let $V$ be the variety generated by any monoid in the variety $A_0^1 \lor Z_6$. If either $A_0 \notin V$ or $Z_6 \notin V$, then the variety $V$ is finitely based.
**Proof.** By Proposition 5.3, the variety \( V \) satisfies the identities (5.2) with \( n = 6 \).

**Case 1:** \( A_0^1 \not\in V \). It follows from Lemmas 2.4 and 5.6 that the variety \( V \) satisfies the identity (5.4). Then \( V \) also satisfies the identity \( xyy \approx x^2y^2 \) since

\[
xyxy \overset{(5.2)}{=} x^2x^6y^6y^2 \overset{(5.4)}{=} x^2x^6y^6y^2 \overset{(5.2b)}{=} x^2y^2.
\]

It is easy to check that \( V \) is finitely based by Condition 7.

**Case 2:** \( Z_6 \not\in V \). Then either \( Z_2 \not\in V \) or \( Z_3 \not\in V \). First assume that \( Z_2 \in V \) and \( Z_3 \in V \). Then by Lemma 2.2(iii), the variety \( V \) satisfies the identity \( x^4 \approx x^2 \). Since

\[
xyxzx \overset{(5.2a)}{=} x^2y^7txy \overset{(5.2b)}{=} x^7y^7txy \overset{(5.1)}{=} x^6y^6txy^{2}x \overset{(5.2a)}{=} x^7y^7tyx \overset{(5.2b)}{=} xytxy,
\]

\( V \) is a subvariety of \( A_0^1 \lor Z_2 \) by Proposition 5.3. If \( A_0^1 \not\in V \), then \( V \) is finitely based by Case 1. If \( A_0^1 \in V \), then \( V = A_0^1 \lor Z_6 \) and \( V \) is finitely based by Proposition 5.3.

Now if \( Z_2 \not\in V \) and \( Z_3 \in V \), then \( V \) is finitely based by a similar argument. Hence it remains to assume that \( Z_2 \not\in V \) and \( Z_3 \not\in V \). By Lemma 2.2(iii), \( V \) satisfies the identity \( x^3 \approx x^2 \); it also satisfies the identity \( xyxxz \approx xyzx \) because

\[
xyxxz \overset{(5.2a)}{=} x^2yzx \overset{(5.2b)}{=} x^7yzx \overset{(5.1)}{=} x^6hy^6tyx \overset{(5.2b)}{=} xyzx.
\]

It is then routine to check that \( V \) is finitely based by Condition 5.

\[\Box\]

5.3. **Proof of Proposition 5.1.** Let \( M \) be any monoid that satisfies the identities (5.1).

Then \( M \) satisfies the identities (5.2a) and (5.2b) with \( n = 6 \). Since

\[
xhytxy \overset{(5.2b)}{=} x^7hy^7txy \overset{(5.2a)}{=} x^6hy^6tyx^2x \overset{(5.1)}{=} x^6hy^6txy^2x \overset{(5.2a)}{=} x^7hy^7txy \overset{(5.2b)}{=} xhytxy,
\]

\( M \) satisfies the identities \( xhytxy \approx xhytxy \). By a symmetrical argument, \( M \) also satisfies the identities \( xyhxy \approx yxhxy \). Consequently, \( M \) satisfies the identities (5.2c) and so belongs to the variety \( A_0^1 \lor Z_6 \) by Proposition 5.3.

Let \( M \) be the variety generated by \( M \). If \( A_0^1, Z_6 \in M \), then \( M = A_0^1 \lor Z_6 \) and the monoid \( M \) is finitely based by Proposition 5.3. If either \( A_0^1 \not\in M \) or \( Z_6 \not\in M \), then \( M \) is finitely based by Lemma 5.7.

6. **On Condition 8**

Let \( p \) denote a fixed prime integer throughout this chapter. It is easily shown that Condition 8 is a special case of the following result.

**Proposition 6.1.** Any semigroup that satisfies the identities

\[
xyx^{p+1} \approx xyx, \quad \text{(6.1a)}
\]
\[
xyxxz \approx xyx^2z, \quad \text{(6.1b)}
\]
\[
xhytxy \approx xhytx \quad \text{(6.1c)}
\]

but violates the identity

\[
h^p(xy)^p \approx h^px^py^p \quad \text{(6.2)}
\]

is finitely based.
The proof of Proposition 6.1 is given in Section 6.4. The semigroup
\[ P_2 = \langle a, b \mid a^2 = ab = a, b^2a = b^2 \rangle \]
of order four plays a central role in this proof. Denote by \( P_2 \) the variety generated by the monoid \( P_2^1 \).

**Remark 6.2.**

(i) It is routine to show that the semigroup \( P_2 \) satisfies an identity \( u \approx v \) if and only if the words \( u \) and \( v \) share the same prefix of length two. It follows that the identity \( xyz \approx xy \) forms a basis for the semigroup \( P_2 \).

(ii) The semigroup \( P_2 \) has appeared in Tishchenko [36] as \( 053 \) and in Volkov [43] as \( L_{3,1} \), while the monoid \( P_2^1 \) has appeared in Edmunds et al. [10] as \( D \).

**6.1. A basis for the variety \( P_2^1 \lor \mathbb{Z}_p \).** In this chapter, a word
\[ w = \prod_{i=1}^{m} (x_i w_i) \quad (6.3) \]
is said to be in **p-canonical form** if \( x_1, \ldots, x_m \) are distinct letters and \( w_1, \ldots, w_m \) are possibly empty words such that

(I) \( \text{ini}(w) = x_1 \cdots x_m \);

(II) \( w_i \in \{x_1^{e_1} \cdots x_i^{e_i} \mid e_1, \ldots, e_i \in \{0, \ldots, p\}\} \);

(III) \( \text{con}(w_i) \cap \text{con}(w_j) = \emptyset \) whenever \( i \neq j \).

It follows that for any \( x \in \text{con}(w) \),

(IV) if \( x \) is nonsimple in \( w \), say with \( \text{occ}(x, w) = e \geq 2 \), then \( w = axbx^{e-1}c \) for some \( a, b, c \in (X \setminus \{x\})^* \).

**Lemma 6.3.** Let \( w \) be any word. Then there exists some word \( w' \) in **p-canonical form** such that the identities (6.1) imply the identity \( w \approx w' \).

**Proof.** It suffices to convert the word \( w \), using the identities (6.1), into a word in **p-canonical form.** The identities (6.1a) and (6.1b) can first be applied to the word \( w \) so that

(a) \( \text{occ}(x, w) \leq p + 1 \) for all \( x \in X \);

(b) \( w \) satisfies (IV).

Suppose that \( \text{ini}(w) = x_1 \cdots x_m \). Then the word \( w \) can be written in the form (6.3) that satisfies (I) with \( \text{con}(w_i) \subseteq \{x_1, \ldots, x_i\} \) for all \( i \). For each \( i \), the letters in \( w_i \) are not first occurrences and so can be ordered by the identities (6.1c) according to their indices. Therefore (II) is satisfied in view of (a). It then follows from (b) that (III) is also satisfied. ■

**Proposition 6.4.** The identities (6.1) constitute a basis for the variety \( P_2^1 \lor \mathbb{Z}_p \).

**Proof.** It is routine to verify that \( P_2^1 \lor \mathbb{Z}_p \) satisfies the identities (6.1). It remains to show that any identity \( w \approx w' \) satisfied by \( P_2^1 \lor \mathbb{Z}_p \) is implied by the identities (6.1). In view of Lemma 6.3, the words \( w \) and \( w' \) can be chosen to be in **p-canonical form.** Since
the submonoid \{a, b^2, 1\} of \(P_2\) is isomorphic to \(L_2\), it follows from Lemma 2.2(i) that \(\text{ini}(w) = \text{ini}(w')\). Therefore

\[
\text{w} = \prod_{i=1}^{m} (x_iw_i) \quad \text{and} \quad \text{w}' = \prod_{i=1}^{m} (x_iw'_i).
\]

Since \(\text{occ}(x, \text{w}), \text{occ}(x, \text{w}') \leq p + 1\) for all \(x \in X\) and the submonoid \(\{b, b^2, 1\}\) of \(P_2\) is isomorphic to \(N_2\), it follows from Lemma 2.2(ii)&(iii) that

(a) \(\text{occ}(x, \text{w}) = \text{occ}(x, \text{w}') \leq p + 1\) for all \(x \in X\).

Suppose that \(\ell\) is the least possible integer such that \(\text{con}(w_\ell) \neq \text{con}(w'_\ell)\). Then

(b) \(\text{con}(w_1) = \text{con}(w'_1), \ldots, \text{con}(w_{\ell-1}) = \text{con}(w'_{\ell-1})\)

and there exists some \(k \leq \ell\) such that \(x_k\) belongs to precisely one of \(w_\ell\) and \(w'_\ell\); by symmetry, it suffices to assume that

(c) \(x_k \in \text{con}(w_\ell)\) and \(x_k \notin \text{con}(w'_\ell)\).

Since \(x_k\) is a nonsimple letter in \(w\), it follows from (a) that \(\text{occ}(x_k, w) = \text{occ}(x_k, w') = e\) for some \(e \in \{2, \ldots, p + 1\}\). By (IV), both \(w\) and \(w'\) are of the form \(\cdots x_k \cdots x^e_k \cdots \cdots\). Within the word \(w\), the factor \(x^e_k\) occurs in \(w_\ell\) by (c). However, within the word \(w'\), it follows from (b) and (c) that the factor \(x^e_k\) does not occur in any of \(w'_1, \ldots, w'_{\ell-1}, w'_\ell\). Therefore the factor \(x^e_k\) must occur in \(w'_j\) for some \(j > \ell\). Hence the words \(w\) and \(w'\) are of the form

\[
w = ax_kb x^e_k c x_{\ell+1} d\quad \text{and} \quad w' = ax_kb' x'_{\ell+1} c' x^e_k d'
\]

with \(x_{\ell+1} \notin \text{con}(abca'b')\). Let \(\varphi\) be the following substitution into the monoid \(P_2\):

\[
x \mapsto \begin{cases} b & \text{if } x = x_k, \\ a & \text{if } x = x_{\ell+1}, \\ 1 & \text{otherwise.} \end{cases}
\]

Then \(w\varphi = b^e((cx_{\ell+1}d)\varphi) = b^2\) and \(w'\varphi = ba((c'x^e_kd')\varphi) = ba\), which implies that the identity \(w \approx w'\) is contradictorily not satisfied by the monoid \(P_2\).

Therefore the integer \(\ell\) does not exist, whence \(\text{con}(w_i) = \text{con}(w'_i)\) for every \(i\). It then follows from (a) and (IV) that \(w_i = w'_i\) for every \(i\). Consequently, the identity \(w \approx w'\) is trivial and is implied by the identities (6.1).

6.2. A basis for the variety \(P_2\)

Proposition 6.5. The identities (6.1) and

\[
x^3 \approx x^2
\]

constitute a basis for the variety \(P_2\).

Proof. Let \(w\) be any word. By Lemma 6.3, the identities (6.1) can be used to convert \(w\) into the word (6.3) in \(p\)-canonical form that satisfies (I), (II), and (III). Since

\[
xyx^2 \approx xyx^{p+1} \approx xyx
\]
any of the exponents $e_1, \ldots, e_i$ from (II) that is nonzero can be reduced by the identities $xyx^2 \approx xyx$ to 1. In this proof, the word $w$ in (6.3) is said to be in $P_{12}^1$-canonical form if it satisfies (I), (III), and

II: $w_i \in \{x_{e_1}^1 \cdots x_{e_i}^i | e_1, \ldots, e_i \in \{0,1\}\}.$

As was just shown, the identities (6.1) and (6.4) convert any word into $P_{12}^1$-canonical form. It is routine to verify that the variety $P_{12}^1$ satisfies the identities (6.1) and (6.4). It remains to show that any identity $w \approx w'$ satisfied by $P_{12}^1$ is implied by the identities (6.1) and (6.4). Convert $w$ and $w'$ into words in $P_{12}^1$-canonical form by the identities (6.1) and (6.4). By an argument similar to the proof of Proposition 6.4, the identity $w \approx w'$ is trivial and hence is satisfied by the variety $P_{12}^1$. ■

Corollary 6.6. The identities $\text{xyzxz} \approx \text{xyxz}$, $\text{xhytxy} \approx \text{xhytyx}$ constitute a basis for the variety $P_{12}^1.$

6.3. An exclusion identity

Lemma 6.7. The identity (6.2) is an exclusion identity for the semigroup $P_{12}^1$ in $P_{12}^1 \lor \mathbb{Z}_p$.

Proof. Let $V$ be any subvariety of $P_{12}^1 \lor \mathbb{Z}_p$ such that $P_{12}^1 \notin V$. First note that the identity $(xy)^p \approx x^p y^p$ (6.5) is not satisfied by the semigroup $P_{12}^1$ since $(ba)^p \neq b^p a^p$. Let $P_{12}$ be the semigroup that is anti-isomorphic to $P_{12}^1$. The semigroups $A_0, B_2, \text{ and } P_{12}$ do not satisfy the identity $xyxzx \approx xyx^2z$ from (6.1b) so that $A_0, B_2, P_{12} \notin V$. It then follows from Volkov [43, Theorem 2.1] that the variety $V$ satisfies the identity (6.5). Therefore the identity (6.5) is an exclusion identity for $P_{12}^1$ in $P_{12}^1 \lor \mathbb{Z}_p$.

Now denote by $\psi$ the substitution $z \mapsto h^p z h^p$ for all $z \in \{x, y\}$. Then it follows from Lee [18, Theorem 2] that $(\text{((xy)^p)} \psi) \approx (x^p y^p) \psi$ is an exclusion identity for $P_{12}^1$ in $P_{12}^1 \lor \mathbb{Z}_p$; it is easy to show that this identity is equivalent to the identity (6.2) within $P_{12}^1 \lor \mathbb{Z}_p$. ■

6.4. Proof of Proposition 6.1. Let $S$ be any semigroup that satisfies the identities (6.1) but violates the identity (6.2), and let $V$ be the variety generated by $S$. Then $V \subseteq P_{12}^1 \lor \mathbb{Z}_p$ by Proposition 6.4 and $P_{12}^1 \in V$ by Lemma 6.7. If $\mathbb{Z}_p \in V$, then $V = P_{12}^1 \lor \mathbb{Z}_p$ and $S$ is finitely based by Proposition 6.4. Therefore assume that $\mathbb{Z}_p \notin V$. By Lemma 2.2(iii), the variety $V$ satisfies the identity (6.4) so that $V \subseteq P_{12}^1$ by Proposition 6.5. Since $P_{12}^1 \in V$, it follows that $V = P_{12}^1$, whence the semigroup $S$ is finitely based by Proposition 6.5.

7. On Condition 9

The main aim of the present chapter is to establish the finite basis property of any semigroup that satisfies the identities $x^4 \approx x^2, \text{ } x^3 y x \approx x y x, \text{ } x y x^2 \approx x^3 y, \text{ } x y x y \approx x y^2 x$ (7.1)
but violates both of the identities
\[ xyxy \approx x^2y^2, \quad (7.2a) \]
\[ xy^3x \approx xyx. \quad (7.2b) \]

Let \( U \) be the variety defined by the identities
\[ xyx^2 \approx x^3y, \quad (7.3a) \]
\[ x^2yx^2 \approx x^2y, \quad (7.3b) \]
\[ xyx^2zx^* \approx xyzx, \quad (7.3c) \]
\[ xhytxy^* \approx xhytyx. \quad (7.3d) \]

7.1. Identities of \( U \). In this chapter, a word
\[ w = x_0^{e_0} \prod_{i=1}^m (x_i^{e_i} w_i) \quad (7.4) \]
is said to be in canonical form if \( x_0, \ldots, x_m \) are distinct letters and \( w_1, \ldots, w_m \) are possibly empty words that satisfy all of the following conditions:

(I) \( \text{ini}(w) = x_0 \cdots x_m; \)
(II) \( w_i \in \{x_0^{f_0} \cdots x_{i-1}^{f_{i-1}} | f_0, \ldots, f_{i-1} \in \{0,1\}\}; \)
(III) \( e_0, \ldots, e_m \in \{1,2,3\}; \)
(IV) if \( e_i = 3 \), then \( x_i \notin \text{con}(w_{i+1} \cdots w_m). \)

Note that \( x_i \notin \text{con}(w_1 \cdots w_i) \) by (II).

Lemma 7.1. Let \( w \) be any word. Then there exists some word \( w' \) in canonical form such that the identities (7.3) imply the identity \( w \approx w' \).

Proof. It suffices to convert the word \( w \), using the identities (7.3), into a word in canonical form. Without loss of generality, assume that \( \text{ini}(w) = x_0 \cdots x_m \). Then the word \( w \) can be written in the form
\[ w = x_0^{e_0} \prod_{i=1}^m (x_i u_i) \quad (7.5) \]
for some \( e_0 \geq 1 \) and some \( u_1, \ldots, u_m \in X^* \) such that \( \text{con}(u_i) \subseteq \{x_0, \ldots, x_i\} \) for all \( i \). For each \( i \), the letters in \( u_i \) are not first occurrences in the word \( w \) so that the identities (7.3d) can be used to permute them in any manner within \( u_i \). Specifically, any occurrence of \( x_i \) in \( u_i \) can be moved to the left and gathered with the \( x_i \) that immediately precedes \( u_i \) in (7.5), and any of the letters \( x_0, \ldots, x_{i-1} \) in \( u_i \) can be ordered according to their indices. The resulting word is of the form (7.4) that satisfies (I) and the condition that \( w_i \in \{x_0^{f_0} \cdots x_{i-1}^{f_{i-1}} | f_0, \ldots, f_{i-1} \geq 0\} \).

Let \( f_0, \ldots, f_{i-1} \geq 0 \) be such that \( w_i = x_0^{f_0} \cdots x_{i-1}^{f_{i-1}} \). Suppose that \( f_j \geq 2 \) for some \( j \in \{0, \ldots, i-1\} \), say \( f_j = 2p + r \) with \( p \geq 1 \) and \( r \in \{0,1\} \). Then the identities (7.3a)
can be used to gather all multiples of \(x_j^2\) in \(w_i\) with the first \(x_j\) in \(w\):

\[
\begin{align*}
\mathbf{w} &= \cdots x_j^{e_j} w_j \cdots x_i^{e_i} \left( x_0^{f_0} \cdots x_{j-1}^{2p+q} x_{j+1}^{f_{j+1}} \cdots x_{i-1}^{f_{i-1}} \right) \cdots \\
&\quad \overset{7.3a}{\approx} \cdots x_j^{e_j+2p} w_j \cdots x_i^{e_i} \left( x_0^{f_0} \cdots x_{j-1}^{f_{j-1}} x_{j+1}^{r} x_{j+1}^{f_{j+1}} \cdots x_{i-1}^{f_{i-1}} \right) \cdots 
\end{align*}
\]

For any other \(k \neq j\), the same argument can be repeated to gather all multiples of \(x_k^2\) in \(w_i\) with the first \(x_k\) in \(w\). Hence (II) is satisfied. It is clear that (III) is satisfied by applying the identity \(x^4 \approx x^2\) from (7.3c). If \(e_\ell = 3\) and \(x_\ell \in \text{con}(w_{\ell+1} \cdots w_m)\), then the identities (7.3c) can be used to reduce the exponent \(e_\ell\) to 1. Therefore (IV) is satisfied.

**Lemma 7.2.** Let \(w\) and \(w'\) be any words in canonical form such that

(a) \(\text{occ}(x, w) \equiv \text{occ}(x, w') \pmod{2}\) for all \(x \in \mathcal{X}\);

(b) \(\text{sim}(w) = \text{sim}(w')\);

(c) \(\text{ini}(w) = \text{ini}(w')\);

(d) \(\text{occ}(x, y, w) \equiv \text{occ}(x, y, w') \pmod{2}\) for all \(x, y \in \mathcal{X}\).

Then the words \(w\) and \(w'\) are identical.

**Proof.** By (c), it can be assumed that \(\text{ini}(w) = \text{ini}(w') = x_0 \cdots x_m\). Since the words \(w\) and \(w'\) are in canonical form,

\[
\begin{align*}
\mathbf{w} &= x_0^{e_0} \prod_{i=1}^{m} (x_i^{e_i} w_i) \\
\mathbf{w}' &= x_0^{e'_0} \prod_{i=1}^{m} (x_i^{e'_i} w'_i)
\end{align*}
\]

for some \(e_i, e'_i \geq 1\) and \(w_i, w'_i \in \mathcal{X}^*\). Since \(\text{occ}(x_m, w) = e_m\) and \(\text{occ}(x_m, w') = e'_m\) by (II), it follows from (III), (a), and (b) that \(e_m = e'_m\) necessarily. Suppose that \(\ell < m\) is the least integer such that \(e_\ell \neq e'_\ell\). By symmetry, it suffices to assume that \(e_\ell > e'_\ell\), whence \((e_\ell, e'_\ell) \in \{(2, 1), (3, 1), (3, 2)\}\) by (III). Since (II) implies that \(\text{occ}(x_\ell, x_{\ell+1}, w) = e_\ell\) and \(\text{occ}(x_\ell, x_{\ell+1}, w') = e'_\ell\), it follows from (d) that \((e_\ell, e'_\ell) = (3, 1)\). Hence (II) and (IV) imply that \(\text{occ}(x_\ell, w) = 3\) and

\[
\begin{align*}
\mathbf{w} &= x_0^{e_0} \left( \prod_{i=1}^{\ell-1} (x_i^{e_i} w_i) \right) x_\ell^3 w_{\ell+1} \prod_{i=\ell+1}^{m} (x_i^{e_i} w_i),
\end{align*}
\]

where \(x_\ell\) does not occur in any of the factors \(w_1, \ldots, w_m\). Since \(\text{occ}(x_\ell, w) = 3\), it follows from (a) and (b) that \(\text{occ}(x_\ell, w')\) is odd and at least three, say \(\text{occ}(x_\ell, w') = k + 1\) for some even \(k \geq 2\). Since \(e'_\ell = 1\), the factor \(w'_\ell\) of \(w'\) is preceded by the first \(x_\ell\) in \(w'\) and the remaining \(k\) occurrences of \(x_\ell\) in \(w'\) are spread out in some \(w'_1, \ldots, w'_k\) with \(\ell < \ell_1 < \cdots < \ell_k \leq m\), that is,

\[
\begin{align*}
\mathbf{w}' &= \cdots x_\ell w'_1 \cdots x_\ell \left( x_\ell \cdots \right) \cdots x_\ell_k \left( \cdots x_\ell \cdots \right) \cdots \\
&= \cdots x_\ell w'_1 \cdots x_\ell \cdots w'_k \cdots
\end{align*}
\]

But now \(\text{occ}(x_\ell, x_{\ell_2}, w') = 2\) and \(\text{occ}(x_\ell, x_{\ell_2}, w) = 3\) contradict (d). Therefore the integer \(\ell\) does not exist, whence \((e_0, \ldots, e_m) = (e'_0, \ldots, e'_m)\).

If \((w_1, \ldots, w_{m-1}) = (w'_1, \ldots, w'_{m-1})\), then it follows from (II) and (a) that \(w_m = w'_m\), whence the identity \(w \approx w'\) is trivial and is satisfied by the variety \(U\). Thus it remains to consider the case when \((w_1, \ldots, w_{m-1}) \neq (w'_1, \ldots, w'_{m-1})\). Then there is a least
integer \( \ell < m \) such that \( w_\ell \neq w'_\ell \), say \( x_r \in \text{con}(w_\ell) \setminus \text{con}(w'_\ell) \) for some \( r < \ell \). Since \( (w_1, \ldots, w_{\ell-1}) = (w'_1, \ldots, w'_{\ell-1}) \), it follows from (II) that

\[
\text{occ}(x_r, x_{\ell+1}, w) = \text{occ}(x_r, x_0^{e_0} \prod_{i=1}^{\ell-1} (x_i^{e_i} w_i)) + \text{occ}(x_r, x_\ell w_\ell) \\
= \text{occ}(x_r, x_0^{e_0} \prod_{i=1}^{\ell-1} (x_i^{e_i} w_i)) + 1,
\]

\[
\text{occ}(x_r, x_{\ell+1}, w') = \text{occ}(x_r, x_0^{e_0} \prod_{i=1}^{\ell-1} (x_i^{e_i} w'_i)) + \text{occ}(x_r, x_\ell w'_\ell) \\
= \text{occ}(x_r, x_0^{e_0} \prod_{i=1}^{\ell-1} (x_i^{e_i} w_i)) + 0,
\]

whence (d) is violated. Consequently, the integer \( \ell \) does not exist, whence the identity \( w \approx w' \) is trivial and is satisfied by the variety \( U \).

Lemma 7.3. The variety \( U \) satisfies an identity \( w \approx w' \) if and only if the four conditions in Lemma 7.2 hold.

Proof. By Lemma 7.1, the words \( w \) and \( w' \) can be chosen to be in canonical form. Suppose that the variety \( U \) satisfies the identity \( w \approx w' \). It follows from Lemma 2.2 that the variety \( U \) contains the monoids \( L_2, N_2 \), and \( Z_2 \) so that (a)--(c) in Lemma 7.2 hold. It is easy to show that the orthodox completely regular monoid

\[
O = \langle a, b \mid a^2 = ab = a, b^2 = 1 \rangle
\]

of order four belongs to the variety \( U \) and so must satisfy the identity \( w \approx w' \). If \( \text{occ}(x, y, w) \neq \text{occ}(x, y, w') \) (mod 2), then letting \( \varphi \) be the substitution

\[
z \mapsto \begin{cases} 
    b & \text{if } z = x, \\
    a & \text{if } z = y, \\
    1 & \text{if } z \in X \setminus \{x, y\},
\end{cases}
\]

into \( O \), the contradiction \( \{w\varphi, w'\varphi\} = \{a, ba\} \) is deduced. Thus (d) in Lemma 7.2 holds.

Conversely, if the four conditions in Lemma 7.2 hold, then the identity \( w \approx w' \) is trivial and so is satisfied by the variety \( U \).

7.2. Identities satisfied by proper subvarieties in \( U \). An identity \( w \approx w' \) deletes to an identity \( u \approx u' \) if, up to letter renaming, the identity \( u \approx u' \) belongs to the system \( w \approx w' \).

Lemma 7.4. Suppose that the identity \( w \approx w' \) deletes to the identity \( u \approx u' \). Then the identities (7.3) and \( w \approx w' \) imply the identity \( h^2 u \approx h^2 u' \).

Proof. Suppose that the identity \( u \approx u' \) is obtained from the identity \( w \approx w' \) by deleting all occurrences of the letters \( x_1, \ldots, x_k \). Denote by \( \varphi \) the substitution \( z \mapsto h^2 \) for all \( z \in X \setminus \{x_1, \ldots, x_k\} \). Since

\[
h^2 u \approx h^2(w \varphi) \approx h^2(w' \varphi) \approx h^2 u',
\]

the identities (7.3) and \( w \approx w' \) imply the identity \( h^2 u \approx h^2 u' \).

}\]
LEMMA 7.5. Any proper subvariety of $U$ satisfies some identity from (7.2).

Proof. Let $W$ be any proper subvariety of $U$. Then the variety $W$ satisfies some identity $w \approx w'$ that is not satisfied by the variety $U$. By Lemma 7.3, at least one of the four conditions in Lemma 7.2 does not hold for the identity $w \approx w'$.

CASE 1: $\text{occ}(x, w) \not\equiv \text{occ}(x, w') \pmod{2}$ for some $x \in \mathcal{X}$. Then the variety $W$ satisfies the identity $\sigma_1 : x^3 \approx x^2$; it also satisfies the identity (7.2a) since

$$\begin{align*}
xyxy &\equiv xy^3 \sigma_1 \equiv xy^2 \sigma_2 \equiv x^3 \sigma_1 \equiv x^2 y^2.
\end{align*}$$

CASE 2: $\text{sim}(w) \not\equiv \text{sim}(w')$.

SUBCASE 2.1: $\text{con}(w) = \text{con}(w')$. Then the identity $w \approx w'$ deletes to the identity $x^m \approx x$ for some $m \geq 2$. It follows from Lemma 7.4 that the variety $W$ satisfies the identity $h^2 x^m \approx h^2 x$ and so also the identity $\sigma_2 : h^2 x^3 \approx h^2 x$. Since

$$\begin{align*}
xy^3 x &\equiv x^3 y^3 \sigma_2 \equiv x^3 y x \sigma_2 \equiv x y x,
\end{align*}$$

the variety $W$ satisfies the identity (7.2b).

SUBCASE 2.2: $\text{con}(w) \not\equiv \text{con}(w')$. Without loss of generality, assume that

$$\text{con}(w) \setminus \text{con}(w') = \{x_1, \ldots, x_m\} \quad \text{and} \quad \text{con}(w') \setminus \text{con}(w) = \{y_1, \ldots, y_n\}$$

for some $m, n \geq 0$. The assumption $\text{con}(w) \not\equiv \text{con}(w')$ implies that $(m, n) \neq (0, 0)$. Then the variety $W$ satisfies the identity $\nu \approx \nu'$ where

$$\begin{align*}
\nu &= w x_1 \cdots x_m y_1 \cdots y_n \quad \text{and} \quad \nu' = w' x_1 \cdots x_m y_1 \cdots y_n.
\end{align*}$$

It is clear that $\text{con}(\nu) = \text{con}(\nu')$ and $\text{sim}(\nu) \not\equiv \text{sim}(\nu')$. Therefore it follows from Sub-case 2.1 that the variety $W$ satisfies the identity (7.2b).

CASE 3: $\text{ini}(w) \not\equiv \text{ini}(w')$. If $\text{con}(w) \not\equiv \text{con}(w')$, say with $x \in \text{con}(w) \setminus \text{con}(w')$, then since the variety $W$ satisfies the identity $x w \approx x w'$ with $\text{sim}(x w) \not\equiv \text{sim}(x w')$, it follows from Case 2 that this variety also satisfies the identity (7.2b). Therefore it can further be assumed that $\text{con}(w) = \text{con}(w')$. The identity $w \approx w'$ then deletes to an identity of the form $x p y^q u \approx y^{p'} x^{q'} u'$ for some $p, q, p', q' \geq 1$ and $u, u' \in \{x, y\}^*$. It follows from Lemma 7.4 that the variety $W$ satisfies the identity $h^2 x p y^q u \approx h^2 y^{p'} x^{q'} u'$. Denote by $\varphi$ the substitution $z \mapsto z^2$ for all $z \in \mathcal{X}$. Since

$$\begin{align*}
(h^2 x p y^q u) \varphi &\approx h^2 x^2 y^2 \quad \text{and} \quad (h^2 y^{p'} x^{q'} u') \varphi \approx h^2 y^{2x^2},
\end{align*}$$

the variety $W$ satisfies the identity $\sigma_4 : h^2 x^2 y^2 \approx h^2 y^{2x^2}$. Since

$$\begin{align*}
xyxy &\equiv x^2 (y xy)^2 \sigma_4 \equiv x^2 y^2 (xy)^2 \sigma_4 \equiv x^2 y^2 x^2 y^2 \sigma_4 \equiv x^2 y^2,
\end{align*}$$

the variety $W$ satisfies the identity (7.2a).

CASE 4: $\text{occ}(x, y, w) \not\equiv \text{occ}(x, y, w') \pmod{2}$ for some $x, y \in \mathcal{X}$. In view of Cases 1 and 3, it suffices to further assume that

(a) $\text{occ}(z, w) \equiv \text{occ}(z, w') \pmod{2}$ for all $z \in \mathcal{X}$;

(b) $\text{ini}(w) = \text{ini}(w')$. 

Minimal non-finitely based monoids

Let \( u \approx u' \) be the identity that is obtained from \( w \approx w' \) by retaining only the letters \( x \) and \( y \) so that \( \text{occ}(x, y, u) = \text{occ}(x, y, w) \) and \( \text{occ}(x, y, u') = \text{occ}(x, y, w') \). By Lemma 7.4, the variety \( W \) satisfies the identity \( h^2u \approx h^2u' \). By (b), there is no loss of generality in assuming that \( \text{ini}(u) = \text{ini}(u') = xy \), whence \( \text{occ}(x, y, u), \text{occ}(x, y, u') \geq 1 \). Let

\[
p = \text{occ}(x, y, u), \quad q = \text{occ}(x, u) - p, \quad p' = \text{occ}(x, y, u'), \quad \text{and} \quad q' = \text{occ}(x, u') - p'.
\]

Then \( p \not \equiv p' \pmod{2} \) by the assumption of the present case and

\[
p + q = \text{occ}(x, u) = \text{occ}(x, w) \equiv \text{occ}(x, w') = \text{occ}(x, u') = p' + q' \pmod{2}
\]

by (a), whence \( q \not \equiv q' \pmod{2} \). Let \( r, s \in \{0, 1\} \) be such that

\[
r + p \equiv 0 \not \equiv 1 \equiv r + p' \pmod{2} \quad \text{and} \quad s + q \equiv 0 \not \equiv 1 \equiv s + q' \pmod{2}.
\]

Denote by \( \varphi \) the substitution

\[
z \mapsto \begin{cases} y^2 & \text{if } z = y, \\ x & \text{if } z = h. \end{cases}
\]

Since

\[
x^r((h^2u)\varphi)x^s \overset{(7.3a)}{\approx} x^{r+2+p}y^{2\text{occ}(y, u)}x^{q+s} \overset{(7.3c)}{\approx} x^2y^2x^2 \overset{(7.3b)}{\approx} \text{occ}(x, u) = p' + q' \pmod{2}
\]

the variety \( W \) satisfies the identity \( (7.2a) \). 

7.3. Proof of Condition 9. Let \( S \) be any semigroup that satisfies the identities \( (7.1) \) but violates both identities in \( (7.2) \). Then it is easy to show that \( S \) satisfies the identities \( (7.3a), (7.3b), \) and \( (7.3c) \). Since

\[
xhtxy \overset{(7.3c)}{\approx} xhty^2xy \overset{(7.3c)}{\approx} xhtyx^2xy \overset{(7.3c)}{\approx} xhty^2x \overset{(7.3c)}{\approx} xhty^3 \overset{(7.3c)}{\approx} xhtytx,
\]

the semigroup \( S \) also satisfies the identities \( (7.3d) \), whence \( S \in U \). Since \( S \) does not satisfy any identity in \( (7.2) \), it follows from Lemma 7.5 that \( S \) must generate the variety \( U \) and is finitely based.

8. The monoid \( A \)

8.1. Main result

**Proposition 8.1.** The identities

\[
\begin{align*}
xhxtx & \approx htx^3, \quad x^4 \approx x^3, \quad (8.1a) \\
xhy^3x & \approx xhxy^3, \quad x^3y^3 \approx y^3x^3, \quad (8.1b) \\
xhtyx & \approx xhtyx, \quad xyhxty \approx yhxhty \quad (8.1c)
\end{align*}
\]
constitute a basis for the monoid \( A \) with the following multiplication table:

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</tr>
</tbody>
</table>

Note that the identity element of the monoid \( A \) is 6.

8.2. A canonical form. In this chapter, a nonsimple word

\[ w = \prod_{i=1}^{m} (s_i w_i) \]  

(8.2)

is said to be in canonical form if all of the following conditions hold:

(I) the letters of \( s_1 \in X^* \) and \( s_2, \ldots, s_m \in X^+ \) are all simple in \( w \);

(II) the letters of \( w_1, \ldots, w_{m-1} \in X^+ \) and \( w_m \in X^* \) are all nonsimple in \( w \);

(III) \( \text{occ}(x, w) \leq 3 \) for all \( x \in X \);

(IV) if \( \text{occ}(x, w) = 3 \), then all three occurrences of \( x \) in \( w \) form a factor \( x^3 \) in some \( w_i \);

(V) if \( x^3 \) is a factor of \( w \), then either \( x^3 \) is a suffix of \( w \) or \( x^3 \) is followed by the first occurrence of some letter of \( w \);

(VI) if \( x^3 y^3 \) is a factor of \( w \), then \( x \) alphabetically precedes \( y \) in \( X \).

Note that (I) and (II) imply that \( \text{con}(s_i) \cap \text{con}(w_j) = \emptyset \) for any \( i \) and \( j \).

Lemma 8.2. Let \( w \) be any nonsimple word. Then there exists some word \( w' \) in canonical form such that the identities (8.1) imply the identity \( w \approx w' \).

Proof. It suffices to convert the nonsimple word \( w \), using the identities (8.1), into a word in canonical form. By gathering adjacent simple letters and adjacent nonsimple letters in \( w \), it is easy to see that \( w \) can be written in the form (8.2) that satisfies (I) and (II). If \( \text{occ}(x, w) = n \geq 3 \), then the identities \( xhtx \approx htx^3 \) from (8.1a) can be used to gather the first \( n-1 \) occurrences of \( x \) with the last \( x \), and the identity \( x^4 \approx x^3 \) from (8.1a) can be used to eliminate all except three occurrences of \( x \). Hence (III) and (IV) are satisfied. It follows that any factor \( x^3 \) in \( w \) that is not a suffix can only be followed by a letter that is either a first or second occurrence. Suppose that some factor \( x^3 \) of \( w \) is followed by the second occurrence of \( y \), that is, \( w = \cdots y \cdots x^3 y \cdots \) with \( \text{occ}(y, w) = 2 \). Then

\[ w = \cdots y \cdots x^3 y \cdots \overset{(8.1b)}{\approx} \cdots y \cdots yx^3 \cdots . \]

If the factor \( x^3 \) in the word \( \cdots y \cdots yx^3 \cdots \) is followed by another second occurrence, say of \( z \), then \( x^3 \) and this \( z \) can be interchanged by the identities (8.1b). This argument can be repeated sufficiently many times until \( x^3 \) is either followed by a first occurrence or a suffix of the entire word. Hence (V) is satisfied. It is easy to see that (VI) is satisfied by applying the identity \( x^3 y^3 \approx y^3 x^3 \) from (8.1b).
8.3. Identities of the monoid \( A \)

**Lemma 8.3.** The monoid \( A \) does not satisfy any of the following identities:

\[
xy \approx yx, \quad xyx \approx x^2 y, \quad xyx \approx yx^2, \quad x^2 y \approx yx^2, \quad x^3 y \approx yx^3,
\]

\[
x^2 y^2 \approx x y x, \quad x^2 y^2 \approx y x y, \quad x^2 y^2 \approx x y^2 x, \quad x^2 y^2 \approx y x^2 y, \quad x^2 y^2 \approx y^2 x^2,
\]

\[
x^3 y^2 \approx y^2 x^3, \quad x^3 y^2 \approx y x^3 y.
\]

**Proof.** The third identity from (8.3a) is violated in \( A \) by the substitution \((x, y) \mapsto (5, 4)\).

The other four identities in (8.3a) are violated in \( A \) by the substitution \((x, y) \mapsto (4, 2)\).

All identities in (8.3b) and (8.3c) are violated in \( A \) by the substitution \((x, y) \mapsto (4, 5)\). \( \square \)

**Lemma 8.4.** Let

\[
w = \prod_{i=1}^{m}(s_i w_i) \quad \text{and} \quad w' = \prod_{i=1}^{m'}(s_i' w_i')
\]

be nonsimple words in canonical form. Suppose that the monoid \( A \) satisfies the identity \( w \approx w' \). Then

(i) \( m = m' \) with \( s_i = s_i' \) and \( w_i \preceq w_i' \) for all \( i \);

(ii) the identity \( w \approx w' \) preserves complete precedence.

**Proof.** (i) Since the submonoid \( \{1, 2, 6\} \) of \( A \) is isomorphic to the monoid \( N_2^3 \), it follows from Lemma 2.2(ii) that \( \text{con}(w) = \text{con}(w') \) and \( \text{sim}(w) = \text{sim}(w') \). Since \( A \) does not satisfy the identity \( x^3 \approx x^2 \), it follows from (III) that \( \text{occ}(x, w) = \text{occ}(x, w') \) for all \( x \in X \). Further, \( A \) does not satisfy any of the identities in (8.3a) so that \( m = m' \) with \( s_i = s_i' \) and \( w_i \preceq w_i' \) for all \( i \).

(ii) Suppose that \( x \preceq_w y \) for some \( x, y \in X \) with \( p = \text{occ}(x, w) = \text{occ}(x, w') \) and \( q = \text{occ}(y, w) = \text{occ}(y, w') \) (so that \( p, q \in \{1, 2, 6\} \) by (III)). Then there are four cases to consider.

**Case 1:** \((p, q) \in \{(1, 1), (2, 2), (2, 1), (2, 1), (1, 3), (3, 1)\} \). Since the monoid \( A \) does not satisfy any of the identities in (8.3a) and (8.3b), it follows that \( x \preceq_w y \).

**Case 2:** \((p, q) = (3, 2) \). Then \( w = \cdots x^3 \cdots y \cdots y \cdots \). If \( x \preceq_w y \), then the word \( w' \) is of the form \( \cdots y \cdots y \cdots x^3 \cdots \) or of the form \( \cdots y \cdots x^3 \cdots y \cdots \), whence the monoid \( A \) contradictorily satisfies some identity from (8.3c). Therefore \( x \preceq_w y \) necessarily.

**Case 3:** \((p, q) = (2, 3) \). Then \( w = a x b y^3 d \) for some \( a, b, c, d \in (X \setminus \{x, y\})^* \). Working toward a contradiction, suppose that \( x \preceq_w y \). If \( y \preceq_w x \), then the monoid \( A \) satisfies the identity \( x^2 y^3 \approx y^3 x^2 \) in (8.3c) and so contradicts Lemma 8.3. Therefore \( y \prec_w x \) and it follows that \( w' = a' x b' y^3 c' x d' \) for some \( a', b', c', d' \in (X \setminus \{x, y\})^* \). By (V), the factor \( c' \) can be neither empty nor of the form \( z_1^3 \cdots z_r^3 \) with \( z_1, \ldots, z_r \in X \). It follows that there exists some letter \( z \) with \( \text{occ}(z, w') \in \{1, 2\} \) such that the first occurrence of \( z \) is in \( c' \). Then \( y \preceq_w z \) so that \( y \preceq_w z \) by Case 2. It follows that \( x \preceq_w z \) and \( x \not\prec_w z \), which is impossible by Case 1.

**Case 4:** \((p, q) = (3, 3) \). Then \( w = a x^3 b y^3 c \) for some \( a, b, c \in (X \setminus \{x, y\})^* \). Working toward a contradiction, suppose that \( x \preceq_w y \). Then \( w' = a' y^3 b' x^3 c' \) for some \( a', b', c' \in X \setminus \{x, y\} \). Working toward a contradiction, suppose that \( x \preceq_w y \). Then \( w' = a' x^3 b' y^3 c' \) for some \( a', b', c' \in X \setminus \{x, y\} \).
identities (8.1). It remains to show that any identity satisfies the identities (8.1) because \( w = ax^3h_1^3 \cdots h_r^3y^3c; \)
(b) \( y \) alphabetically precedes \( x \) because \( w' = a'y^3t_1^3 \cdots t_s^3x^3c'. \)
Therefore by symmetry, assume that \( b \) contains the first occurrence of some letter \( z \) with \( \text{occ}(z, w) \in \{1, 2\} \). Then \( x \preceq_w z \) and so \( x \preceq_w z \) by Case 1 or 2. It follows that \( y \not\preceq_w z \) and \( y \preceq_{w'} z \), contradicting Case 1 or 2. 

8.4. Proof of Proposition 8.1. It is routine to verify that the monoid \( A \) satisfies the identities (8.1). It remains to show that any identity \( w \approx w' \) satisfied by \( A \) is implied by the identities (8.1). Since \( A \) is noncommutative and its submonoid \( \{1, 2, 6\} \) is isomorphic to the monoid \( N^2_2 \), it follows from Lemma 2.3 that the identity \( w \approx w' \) is trivial if either \( w \) or \( w' \) is a simple word. Hence assume that the words \( w \) and \( w' \) are both nonsimple.

In view of Lemma 8.2 these words can be assumed to be in canonical form. Therefore by Lemma 8.4(i),
\[
w = \prod_{i=1}^{m}(s_iw_i) \quad \text{and} \quad w' = \prod_{i=1}^{m}(s_iw'_i)
\]
with \( w_i \cong w'_i \) for all \( i \). By Lemma 8.4(ii), the identity \( w \approx w' \) preserves complete precedence. If \( w_i = w'_i \) for all \( i \), then the identity \( w \approx w' \) is implied by the identities (8.1). Therefore suppose that \( \ell \) is the least integer such that \( w_\ell \neq w'_\ell \). Let
\[
p = \left( \prod_{i=1}^{\ell-1}(s_iw_i) \right)s_\ell, \quad r = \prod_{i=\ell+1}^{m}(s_iw_i), \quad \text{and} \quad r' = \prod_{i=\ell+1}^{m}(s_iw'_i).
\]
Then \( w = pw_\ell r \) and \( w' = pw'_\ell r' \) with \( r \cong r' \). It is routine to show that the identity \( pw_\ell r \approx pw'_\ell r' \) preserves complete precedence. Hence by Lemma 2.5 the identities (8.1c) imply the identity \( pw_\ell r \approx pw'_\ell r', \) that is,
\[
w = s_1w_1 \cdots s_\ell w_\ell \cdots s_mw_m \cong s_1w_1 \cdots s_\ell w'_\ell \cdots s_mw_m.
\]
It is easy to see that the same argument can be repeated on \( w_{\ell+1}, \ldots, w_m \) to obtain
\[
w \cong s_1w_1 \cdots s_{\ell-1}w_{\ell-1} s_\ell w'_\ell s_{\ell+1}w'_{\ell+1} \cdots s_mw_m = w'.
\]
Consequently, the identities (8.1) imply the identity \( w \approx w' \).

9. The monoid \( B \)

9.1. Main result

Proposition 9.1. The identities
\[
\begin{align*}
  xhxtx & \approx htx, \quad (9.1a) \\
xhytxy & \approx xhytx, \quad xhxyty \approx xhyty, \quad xyhxyt \approx yxhxyt \quad (9.1b)
\end{align*}
\]
constitute a basis for the monoid $B$ with the following multiplication table:

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Note that the identity element of the monoid $B$ is 6.

9.2. A canonical form. In this chapter, a nonsimple word

$$w = \prod_{i=1}^{m} (s_i w_i)$$

(9.2)

is said to be in canonical form if

(I) the letters of $s_1 \in \mathcal{X}^*$ and $s_2, \ldots, s_m \in \mathcal{X}^+$ are all simple in $w$;

(II) the letters of $w_1, \ldots, w_{m-1} \in \mathcal{X}^+$ and $w_m \in \mathcal{X}^*$ are all nonsimple in $w$;

(III) for each $i$, the letters of $w_i$ are in alphabetical order;

(IV) $\operatorname{occ}(x, w) \leq 2$ for all $x \in \mathcal{X}$.

Note that (I) and (II) imply that $\operatorname{con}(s_i) \cap \operatorname{con}(w_j) = \emptyset$ for any $i$ and $j$.

Lemma 9.2. Let $w$ be any nonsimple word. Then there exists some word $w'$ in canonical form such that the identities (9.1) imply the identity $w \approx w'$.

Proof. It suffices to convert the nonsimple word $w$, using the identities (9.1), into a word in canonical form. By gathering adjacent simple letters and adjacent nonsimple letters in $w$, it is easy to see that $w$ can be written in the form (9.2) that satisfies (I) and (II). For each $i$, since the letters in $w_i$ are nonsimple in $w$, they can be ordered by the identities (9.1b) so that (III) is satisfied. For any letter $x$ such that $\operatorname{occ}(x, w) \geq 3$, the identities (9.1a) can be used to eliminate all except the last two occurrences of $x$ from $w$ so that (IV) is satisfied.

9.3. Proof of Proposition 9.1. It is routine to verify that the monoid $B$ satisfies the identities (9.1). It remains to show that any identity $w \approx w'$ satisfied by $B$ is implied by the identities (9.1). Since $B$ is noncommutative and its submonoid $\{1, 2, 6\}$ is isomorphic to $\mathbb{N}_2$, it follows from Lemma 2.3 that the identity $w \approx w'$ is trivial if either $w$ or $w'$ is a simple word. Hence assume that the words $w$ and $w'$ are both nonsimple. In view of Lemma 9.2, these words can be assumed to be in canonical form, say

$$w = \prod_{i=1}^{m} (s_i w_i) \quad \text{and} \quad w' = \prod_{i=1}^{m'} (s'_i w'_i).$$

It follows from Lemma 2.2(ii) that $\operatorname{con}(w) = \operatorname{con}(w')$ and $\operatorname{sim}(w) = \operatorname{sim}(w')$. Further, it is routine to show that the monoid $B$ does not satisfy any of the following identities:

$$xy \approx yx, \quad xyx \approx x^2 y, \quad xyx \approx yx^2, \quad x^2 y \approx yx^2.$$
It follows that \( m = m' \) with \( s_i = s'_i \) and \( w_i = w'_i \) for all \( i \). The identity \( w \approx w' \) is thus trivial and implied by the identities (9.1).

10. The monoids \( \mathcal{D} \) and \( \mathcal{H} \)

10.1. Main result

**Proposition 10.1.** The identities

\[
\begin{align*}
    xyx^2 & \approx x^3y, \\
    xyx^2xz & \approx xyzx, \\
    xhxyty & \approx xhxytx, \\
    xhxytw & \approx xhxytx
\end{align*}
\]

constitute a basis for the monoids \( \mathcal{D} \) and \( \mathcal{H} \) with the following multiplication tables:

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Note that the identity elements of the monoids \( \mathcal{D} \) and \( \mathcal{H} \) are both 5.

10.2. A canonical form. In this chapter, a word

\[ w = \prod_{i=1}^{m} (x_i w_i) \]

is said to be in canonical form if \( x_1, \ldots, x_m \) are distinct letters and \( w_1, \ldots, w_m \) are possibly empty words that satisfy all of the following conditions:

(I) \( \text{ini}(w) = x_1 \cdots x_m \);

(II) \( w_i \in \{x_1^{e_1} \cdots x_i^{e_i} | e_1, \ldots, e_{i-1} \in \{0, 1\}, e_i \in \{0, 1, 2\}\}; \)

(III) if \( \text{occ}(x_i, w_i) = 2 \), then \( x_i \notin \text{con}(w_{i+1} \cdots w_m) \);

(IV) if \( w_i \neq \emptyset \), then \( x_{i+1} \) is simple in \( w \).

Note that \( x_i \notin \text{con}(w_1 \cdots w_{i-1}) \) by (II).

**Lemma 10.2.** Let \( w \) be any word. Then there exists some word \( w' \) in canonical form such that the identities (10.1) imply the identity \( w \approx w' \).

**Proof.** It suffices to convert the word \( w \), using the identities (10.1), into a word in canonical form. Observe that (I), (II), and (III) are identical to (I), (II), and (III) in Section 7. Since the identities (10.1) contain the identities (7.3), it follows from Lemma 7.1 that by applying the identities (10.1), the word \( w \) can be converted into the word in (10.2) that satisfies (I), (II), and (III).
Suppose that \( w_i \neq \emptyset \) and \( x_{i+1} \) is nonsimple in \( w \). Let \( e_1, \ldots, e_{i-1}, f_1, \ldots, f_i \in \{ 0, 1 \} \) and \( e_i, f_{i+1} \in \{ 0, 1, 2 \} \) be such that \( w_i = x_1^{e_1} \cdots x_i^{e_i} \) and \( w_{i+1} = x_1^{f_1} \cdots x_i^{f_i} x_{i+1}^{f_{i+1}} \). Since the letters in \( w_i \) and \( w_{i+1} \) are not first occurrences in \( w \),

\[
\begin{align*}
\mathbf{w} &= \cdots x_i \mathbf{w}_i \ x_{i+1} \mathbf{w}_{i+1} \cdots \\
&= \cdots x_i (x_1^{e_1} \cdots x_i^{e_i}) x_{i+1} (x_1^{f_1} \cdots x_i^{f_i} x_{i+1}^{f_{i+1}}) \cdots \\
&\approx 10.1c \cdots x_i x_{i+1} (x_1^{e_1} \cdots x_i^{e_i})(x_1^{f_1} \cdots x_i^{f_i} x_{i+1}^{f_{i+1}}) \cdots \\
&\approx 10.1d \cdots x_i x_{i+1} x_1^{e_1+f_1} \cdots x_i^{e_i+f_i} x_{i+1}^{f_{i+1}} \cdots .
\end{align*}
\]

Hence the factor \( w_i \) is combined with the factor \( w_{i+1} \). This procedure can be repeated on any nonempty factor \( w_j \) that precedes a nonsimple letter \( x_{j+1} \). Hence the word \( w \) can be converted into a word \( w' \) that satisfies (IV). However, the word \( w' \) may no longer satisfy (II) since, for example, some of the exponents \( e_1 + f_1, \ldots, e_i + f_i \) in (10.3) may not be in \( \{ 0, 1 \} \). The arguments in the proof of Lemma 7.1 can then be used to convert the word \( w' \) into a word that satisfies (II). ■

10.3. Identities of the monoids \( D \) and \( H \)

**Lemma 10.3.** Let \( M \in \{ D, H \} \) and let \( w \approx w' \) be any identity satisfied by the monoid \( M \). Then

(i) \( \text{ini}(w) = \text{ini}(w') \) (so that \( \text{con}(w) = \text{con}(w') \));

(ii) \( \text{sim}(w) = \text{sim}(w') \);

(iii) \( \text{occ}(x, w) \equiv \text{occ}(x, w') \pmod{2} \) for all \( x \in \mathcal{X} \).

Further, if \( x \in \text{con}(w) = \text{con}(w') \) and \( y \in \text{sim}(w) = \text{sim}(w') \), then

(iv) \( \text{occ}(x, y, w) \equiv \text{occ}(x, y, w') \pmod{2} \).

**Proof.** The submonoids \( \{ 1, 4, 5 \}, \{ 1, 2, 5 \} \) and \( \{ 5, 6 \} \) of \( M \) are isomorphic to \( L_2, N_2^1 \), and \( \mathbb{Z}_2 \), respectively. Therefore parts (i), (ii), and (iii) follow from Lemma 2.2. Suppose that \( \text{occ}(x, y, w) \not\equiv \text{occ}(x, y, w') \pmod{2} \) for some \( x \in \text{con}(w) = \text{con}(w') \) and \( y \in \text{sim}(w) = \text{sim}(w') \), say \( \text{occ}(x, y, w) = 2p + 1 \) is odd and \( \text{occ}(x, y, w') = 2q \) is even. By part (iii), there exists some \( r \in \{ 1, 2 \} \) such that \( \text{occ}(x, w) + r \equiv 0 \equiv \text{occ}(x, w') + r \pmod{2} \). Denote by \( \varphi \) the following substitution into the monoid \( M \):

\[
\begin{cases}
6 & \text{if } z = x, \\
2 & \text{if } z = y, \\
5 & \text{otherwise}.
\end{cases}
\]

Then

\[
(x w r)^\varphi = 6^{2p+1} \cdot 2 \cdot 6^{\text{occ}(x, w) - (2p+1)+r} = 6 \cdot 2 \cdot 6 = 3,
\]

\[
(x w' r)^\varphi = 6^{2q} \cdot 2 \cdot 6^{\text{occ}(x, w') - 2q + r} = 5 \cdot 2 \cdot 5 = 2,
\]

which is impossible. Hence part (iv) holds. ■

10.4. Proof of Proposition 10.1. Let \( M \in \{ D, H \} \). It is routine to verify that the monoid \( M \) satisfies the identities (10.1). It remains to show that any identity \( w \approx w' \) satisfied by \( M \) is implied by the identities (10.1). In view of Lemma 10.2, the words
w and w' can be assumed to be in canonical form. It follows from Lemma 10.3(i) that
\[ w = \prod_{i=1}^{m}(x_{i}w_{i}) \quad \text{and} \quad w' = \prod_{i=1}^{m}(x_{i}w'_{i}). \]

First assume that \((w_{1}, \ldots, w_{m-1}) = (w'_{1}, \ldots, w'_{m-1})\). Then by (II), there exist exponents \(e_{1}, e'_{1}, \ldots, e_{m-1}, e'_{m-1} \in \{0, 1\}\) such that \(w_{m} = x_{1}^{e_{1}} \cdots x_{m}^{e_{m}}\) and \(w'_{m} = x_{1}^{e'_{1}} \cdots x_{m}^{e'_{m}}\). Hence \((e_{1}, \ldots, e_{m-1}) = (e'_{1}, \ldots, e'_{m-1})\) by Lemma 10.3(iii). Since \(\text{occ}(x_{m}, w) = 1 + e_{m}\) and \(\text{occ}(x_{m}, w') = 1 + e'_{m}\), it follows from Lemma 10.3(ii)\&(iii) that \(e_{m} = e'_{m}\). Therefore \(w_{m} = w'_{m}\). The identity \(w \approx w'\) is thus trivial and implied by the identities (10.1).

It remains to assume that \(w_{\ell} \neq w'_{\ell}\) for some least possible \(\ell < m\), say
\[ w_{\ell} = x_{1}^{e_{1}} \cdots x_{\ell}^{e_{\ell}} \quad \text{and} \quad w'_{\ell} = x_{1}^{e'_{1}} \cdots x_{\ell}^{e'_{\ell}} \]
for some \(e_{1}, e'_{1}, \ldots, e_{\ell-1}, e'_{\ell-1} \in \{0, 1\}\) and \(e_{\ell}, e'_{\ell} \in \{0, 1, 2\}\) with \((e_{1}, \ldots, e_{\ell}) \neq (e'_{1}, \ldots, e'_{\ell})\).

By symmetry, it suffices to assume that
(a) \(e_{j} > e'_{j}\)
for some \(j\). Then \(w_{\ell} \neq \emptyset\) so that by (IV), the letter \(x_{\ell+1}\) is simple in \(w\). Hence \(x_{\ell+1}\) is also simple in \(w'\) by Lemma 10.3(ii). Since
\[ \text{occ}(x_{j}, x_{\ell+1}, w) = \text{occ}(x_{j}, x_{1}w_{1} \cdots x_{\ell-1}w_{\ell-1}x_{\ell}x_{\ell+1}) + \text{occ}(x_{j}, w_{\ell}) \]
\[ = \text{occ}(x_{j}, x_{1}w_{1} \cdots x_{\ell-1}w_{\ell-1}x_{\ell}x_{\ell+1}) + e_{j}, \]
\[ \text{occ}(x_{j}, x_{\ell+1}, w') = \text{occ}(x_{j}, x_{1}w'_{1} \cdots x_{\ell-1}w'_{\ell-1}x_{\ell}x_{\ell+1}) + \text{occ}(x_{j}, w'_{\ell}) \]
\[ = \text{occ}(x_{j}, x_{1}w_{1} \cdots x_{\ell-1}w_{\ell-1}x_{\ell}x_{\ell+1}) + e'_{j}, \]
it follows from Lemma 10.3(iv) that
(b) \(e_{j} \equiv e'_{j} \pmod{2}\).

By (II), the conditions (a) and (b) imply that \(j = \ell\) with \(e_{\ell} = 2\) and \(e'_{\ell} = 0\). It follows from (III) that \(x_{\ell} \not\in \text{con}(w_{\ell+1} \cdots w_{m})\), whence \(\text{occ}(x_{\ell}, w) = \text{occ}(x_{\ell}, x_{\ell}w_{\ell}) = 3\). Therefore \(\text{occ}(x_{\ell}, w') \in \{3, 5, \ldots\}\) by Lemma 10.3(ii)\&(iii). Now \(e'_{\ell} = 0\) implies that \(x_{\ell} \not\in \text{con}(w'_{\ell})\), whence \(\text{occ}(x_{\ell}, w') = 1 + \text{occ}(x_{\ell}, w'_{\ell+1} \cdots w'_{m})\). Therefore there exist least possible integers \(p\) and \(q\) such that \(\text{occ}(x_{\ell}, w'_{p}) = \text{occ}(x_{\ell}, w'_{q}) = 1\) and \(\ell < p < q\). Then \(w'_{p} \neq \emptyset\) so that by (IV), the letter \(x_{p+1}\) is simple in \(w'\). Hence \(x_{p+1}\) is also simple in \(w\) by Lemma 10.3(ii). Now \(\text{occ}(x_{\ell}, x_{p+1}, w) = 3\) and \(\text{occ}(x_{\ell}, x_{p+1}, w') = 2\) so that Lemma 10.3(iv) is violated. Consequently, the assumption \(w_{\ell} \neq w'_{\ell}\) is impossible.

11. The monoid \(\mathcal{E}\)

11.1. Main result

Proposition 11.1. The identities
\[ x^{2}y \approx yx^{2}, \quad (11.1a) \]
\[ xhytxy \approx xhtyxt, \quad xhxyty \approx xhyt, \quad xyhxty \approx ythxty, \quad (11.1b) \]
\[ x^{2}hxytx \approx hxtyx \quad (11.1c) \]
Minimal non-finitely based monoids

constitute a basis for the monoid $E$ with the following multiplication table:

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Note that the identity element of the monoid $E$ is 5.

11.2. A canonical form. In this chapter, a nonsimple word

$$w = w_0 \prod_{i=1}^{m} (s_i w_i)$$

(11.2)

with nonsimple letters $x_1, \ldots, x_r$ is said to be in canonical form if all of the following conditions hold:

(I) the letters of $s_1 \in X^*$ and $s_2, \ldots, s_m \in X^+$ are all simple in $w$;

(II) $w_1, \ldots, w_m \in \{ x_1^{f_1} \cdots x_r^{f_r} \mid f_1, \ldots, f_r \in \{0,1\} \}$ with $w_1, \ldots, w_{m-1} \neq \emptyset$;

(III) $w_0 = x_1^{e_1} \cdots x_r^{e_r}$ for some $e_1, \ldots, e_r \in \{0,2,4\}$;

(IV) if $\text{occ}(x_i, w_1 \cdots w_m) \in \{1,2\}$, then $e_i \in \{0,2\}$;

(V) if $\text{occ}(x_i, w_1 \cdots w_m) \geq 3$, then $e_i = 0$.

Note that $\text{con}(s_i) \cap \text{con}(w_j) = \emptyset$ for any $i$ and $j$.

Lemma 11.2. Let $w$ be any nonsimple word. Then there exists some word $w'$ in canonical form such that the identities (11.1) imply the identity $w \approx w'$.

Proof. It suffices to convert the nonsimple word $w$, using the identities (11.1), into a word in canonical form. Let $x_1, \ldots, x_r$ be the nonsimple letters of $w$. By gathering adjacent simple letters and adjacent nonsimple letters, the word $w$ can be written in the form $w = \prod_{i=1}^{m} (s_i w_i)$ where $w_1, \ldots, w_{m-1} \in \{ x_1, \ldots, x_r \}^+$, $w_m \in \{ x_1, \ldots, x_r \}^*$, and the factors $s_1, \ldots, s_m$ satisfy (I). Since the letters in each $w_i$ are nonsimple in $w$, they can be ordered by the identities (11.1b) so that $w_i = x_1^{e_1} \cdots x_r^{e_r}$ for some $e_1, \ldots, e_r \geq 0$. Any factor $x_j^2$ can be moved by the identity (11.1a) to the left of $s_1$ so that the word $w$ is of the form (11.2) that satisfies (I) and (II), with $w_0 = x_1^{e_1} \cdots x_r^{e_r}$ for some $e_1, \ldots, e_r \in \{0,2,4\}$. It is easy to show that (III), (IV), and (V) are satisfied by applying the identities (11.1c).

11.3. Identities of the monoid $E$

Lemma 11.3. Let $w \approx w'$ be any identity satisfied by the monoid $E$. Then

(i) $\text{con}(w) = \text{con}(w')$ and $\text{sim}(w) = \text{sim}(w')$;

(ii) $\text{occ}(x, w) \equiv \text{occ}(x, w') \pmod{2}$ for all $x \in X$.

Further, if $x \in \text{con}(w) = \text{con}(w')$ and $y \in \text{sim}(w) = \text{sim}(w')$, then

(iii) $\text{occ}(x, y, w) \equiv \text{occ}(x, y, w') \pmod{2}$.
Proof. Since the submonoids \{1, 2, 5\} and \{5, 6\} of \(E\) are isomorphic to \(N_2^1\) and \(\mathbb{Z}_2\) respectively, parts (i) and (ii) follow from Lemma 2.2. To establish part (iii), suppose that \(\text{occ}(x, y, w) \neq \text{occ}(x, y, w')\) (mod 2) for some \(x \in \text{con}(w) = \text{con}(w')\) and some \(y \in \text{sim}(w) = \text{sim}(w')\), say \(\text{occ}(x, y, w)\) is even and \(\text{occ}(x, y, w')\) is odd. Denote by \(\varphi\) the following substitution into the monoid \(E\):

\[
z \mapsto \begin{cases} 
6 & \text{if } z = x, \\
3 & \text{if } z = y, \\
5 & \text{otherwise}.
\end{cases}
\]

Then \(w\varphi \in 5 \cdot 3 \cdot \{5, 6\} = \{3\}\) and \(w'\varphi \in 6 \cdot 3 \cdot \{5, 6\} = \{4\}\), which is impossible.

11.4. Proof of Proposition 11.1. It is routine to verify that the monoid \(E\) satisfies the identities (11.1). It remains to show that any identity \(w \approx w'\) satisfied by \(E\) is implied by the identities (11.1). Since \(E\) is noncommutative and its submonoid \(\{1, 2, 5\}\) is isomorphic to \(N_2^1\), it follows from Lemma 2.3 that the identity \(w \approx w'\) is trivial if either \(w\) or \(w'\) is a simple word. Hence assume that they are both nonsimple. In view of Lemma 11.2 these words can be assumed to be in canonical form, say

\[
w = w_0 \prod_{i=1}^{m}(s_i w_i) \quad \text{and} \quad w' = w'_0 \prod_{i=1}^{m'}(s'_i w'_i).
\]

It follows from Lemma 11.3(i) that \(\text{con}(w) = \text{con}(w')\) and \(\text{sim}(w) = \text{sim}(w')\).

Case 1: \(\text{sim}(w) = \text{sim}(w') = \emptyset\). Then \(m = m' = 1\) with \(s_1 = s'_1 = \emptyset\) so that \(w = w_0 w_1\) and \(w' = w'_0 w'_1\). For each \(x \in \mathcal{X}\), since \(\text{occ}(x, w) \equiv \text{occ}(x, w_1)\) (mod 2) and \(\text{occ}(x, w') \equiv \text{occ}(x, w'_1)\) (mod 2) by (III), it follows from Lemma 11.3(ii) that \(\text{occ}(x, w_1) \equiv \text{occ}(x, w'_1)\) (mod 2). Therefore \(w_1 = w'_1\) by (II). It is then easy to deduce from (III), (IV), (V), and Lemma 11.3(ii) that \(w_0 = w'_0\). Hence the identity \(w \approx w'\) is trivial and is implied by the identities (11.1).

Case 2: \(\text{sim}(w) = \text{sim}(w') \neq \emptyset\). Since the monoid \(E\) is noncommutative, it follows that \(s_1 \cdots s_m = s'_1 \cdots s'_m\). It is first shown that the negation of either \(w_m = w'_m\) or \(s_m = s'_m\) leads to a contradiction.

Subcase 2.1: \(w_m \neq w'_m\). By symmetry, it suffices to assume that \(x \in \text{con}(w_m) \setminus \text{con}(w'_m)\). Then \(\text{occ}(x, w_m) = 1\) by (II) and \(\text{occ}(x, w'_m) = 0\). Let \(z\) be the last letter of \(s_m\) and \(s'_m\). Since \(\text{occ}(x, w) \equiv \text{occ}(x, w')\) (mod 2) by Lemma 11.3(ii), it follows that

\[
\text{occ}(x, z, w) = \text{occ}(x, w) - \text{occ}(x, w_m) \neq \text{occ}(x, w') - \text{occ}(x, w'_m) = \text{occ}(x, z, w') \pmod{2},
\]

whence Lemma 11.3(iii) is contradicted.

Subcase 2.2: \(s_m \neq s'_m\). By symmetry, it suffices to assume that \(|s_m| > |s'_m|\). Since \(s_1 \cdots s_m = s'_1 \cdots s'_m\), there exists some \(s \in \text{con}(s_m) \cap \text{con}(s'_m)\). It follows from (II) that \(\text{occ}(x, w'_{m'-1}) = 1\) for some \(x \in \mathcal{X}\). Hence \(\text{occ}(x, s, w) = \text{occ}(x, w) - \text{occ}(x, w_m)\) and

\[
\text{occ}(x, s, w') = \text{occ}(x, w') - \text{occ}(x, w'_{m'-1}) - \text{occ}(x, w'_m) = \text{occ}(x, w') - 1 - \text{occ}(x, w'_m).
\]
Now since \( \text{occ}(x, w) \equiv \text{occ}(x, w') \pmod{2} \) by Lemma 11.3(ii) and \( w_m = w'_m \) as established in Subcase 2.1, it follows that \( \text{occ}(x, s, w) \not\equiv \text{occ}(x, s, w') \pmod{2} \), contradicting Lemma 11.3(iii).

Arguments similar to Subcases 2.1 and 2.2 can be repeated to successively show that \( w_{m-j} = w'_{m'-j} \) and \( s_{m-j} = s'_{m'-j} \) for each \( j \geq 1 \), whence \( \prod_{i=1}^{m}(s_i w_i) = \prod_{i=1}^{m'}(s'_i w'_i) \). It is then easy to deduce from (III), (IV), and (V) that \( w_0 = w'_0 \). Hence the identity \( w \approx w' \) is trivial and is implied by the identities (11.1).

12. The monoid \( F \)

12.1. Main result

**Proposition 12.1.** The identities

\[
\begin{align*}
xhx^2tx^* & \approx xhtx, \quad (12.1a) \\
xhyx^2ty^* & \approx xhx^2ty, \quad (12.1b) \\
xhytxy^* & \approx xhytxy \quad (12.1c)
\end{align*}
\]

constitute a basis for the monoid \( F \) with the following multiplication table:

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Note that the identity element of the monoid \( F \) is 5.

12.2. A canonical form. In this chapter, a word

\[
w = \prod_{i=1}^{m}(x_i w_i)
\]

is said to be in *canonical form* if \( x_1, \ldots, x_m \) are distinct letters and \( w_1, \ldots, w_m \) are possibly empty words that satisfy all of the following conditions:

(I) initial \( \text{ini}(w) = x_1 \cdots x_m \);
(II) \( w_i \in \{ x_1^{e_1} \cdots x_i^{e_i} \mid e_1, \ldots, e_i \in \{0, 1, 2\} \} \);
(III) if \( \text{occ}(x, w_i) = 2 \), then \( x \notin \text{con}(w_{i+1} \cdots w_m) \);
(IV) if \( \text{occ}(x_j, w_i) = 2 \) for some \( j < i \), then \( x_i \) is simple in \( w \).

Note that \( x_i \notin \text{con}(w_1 \cdots w_{i-1}) \) by (II).

**Lemma 12.2.** Let \( w \) be any word. Then there exists some word \( w' \) in canonical form such that the identities (12.1) imply the identity \( w \approx w' \).
Proof. It suffices to convert the word $w$, using the identities (12.1), into a word in canonical form. Without loss of generality, assume that \( \text{ini}(w) = x_1 \cdots x_m \). Then $w$ can be written in the form (12.2) for some $w_1, \ldots, w_m \in X^*$ such that $\text{con}(w_i) \subseteq \{x_1, \ldots, x_i\}$ for all $i$. For each $i$, the letters in $w_i$ are not first occurrences in $w$ so that the identities (12.1c) can be used to permute them in any manner within $w_i$. Hence for each $i$, the factor $w_i$ is of the form $x_1^{e_1} \cdots x_i^{e_i}$ where $e_1, \ldots, e_i \geq 0$. The letters $x_1, \ldots, x_i$ in $w_i$ are not first occurrences in $w$ so that if $e_j \geq 3$ for some $j \leq i$, then the identities (12.1a) can be used to reduce $e_j$ to a number in $\{1, 2\}$. Therefore (II) is satisfied by every $w_i$.

If $\text{occ}(x_j, w_i) = 2$ for some $j \leq i$ and $x_j \in \text{con}(w_{i+1} \cdots w_m)$, say $x_j \in \text{con}(w_k)$ for some $k > i$, then

$$w = x_j \cdots x_{k} \cdots x_j \cdots$$

and the identities (12.1a) can be used to eliminate the factor $x_j^2$ from $w_i$. Hence (III) is satisfied by $w$.

It is clear that the factor $w_1$ satisfies (IV) vacuously. Suppose that $\ell > 1$ and that the factors $w_{\ell+1}, \ldots, w_m$ have been converted to words that satisfy (IV). It suffices to convert the factor $w_{\ell}$ into a word that satisfies (IV). Let $e_1, \ldots, e_\ell \in \{0, 1, 2\}$ be such that $w_\ell = x_1^{e_1} \cdots x_\ell^{e_\ell}$. Suppose that $\text{occ}(x_j, w_\ell) = 2$ for some $j < \ell$ and that $x_j$ is nonsimple in $w$. Then

$$w = \cdots x_j \cdots x_{\ell-1} w_{\ell-1} \cdot x_\ell \cdot x_1^{e_1} \cdots x_{j}^{e_j} \cdot x_{j+1}^{2} x_{j+1}^{e_{j+1}} \cdots x_{\ell}^{e_\ell} \cdots$$

for some $r \geq 0$ such that $e_{\ell} + r > 0$. Now since the letters in $w_\ell$ are not first occurrences,

$$w = \cdots x_j \cdots x_{\ell-1} w_{\ell-1} \cdot x_\ell \cdot x_1^{e_1} \cdots x_{j-1}^{e_{j-1}} x_j^{2} x_{j+1}^{e_{j+1}} \cdots x_{\ell}^{e_\ell} \cdots$$

(12.1c)

$$\approx \cdots x_j \cdots x_{\ell-1} w_{\ell-1} \cdot x_\ell x_j^{2} \cdot x_1^{e_1} \cdots x_{j-1}^{e_{j-1}} x_j^{e_{j+1}} \cdots x_{\ell}^{e_\ell} \cdots$$

(12.1b)

$$\approx \cdots x_j \cdots x_{\ell-1} (w_{\ell-1} x_j^{2}) \cdot x_\ell x_1^{e_1} \cdots x_{j-1}^{e_{j-1}} x_j^{e_{j+1}} \cdots x_{\ell}^{e_\ell} \cdots$$

(12.3)

Since the letters in $w_{\ell-1}$ are not first occurrences, the factor $x_j^2$ in (12.3) can be moved by the identities (12.1c) to the left until it is combined with any $x_j$ in $w_{\ell-1}$. (If this results in $w_{\ell-1}$ containing more than two occurrences of $x_j$, then the identities (12.1a) can be used to eliminate $x_j^2$ from $w_{\ell-1}$.) Hence the factor $x_j^2$ is eliminated from the factor $w_\ell$. The same argument can be repeated to eliminate any square factors from $w_\ell$. Therefore the factor $w_\ell$ can be converted into a word that satisfies (IV). \[ \square \]

12.3. Identities of the monoid $\mathcal{F}$

Lemma 12.3. Let $w \approx w'$ be any identity satisfied by the monoid $\mathcal{F}$. Then

(i) $\text{ini}(w) = \text{ini}(w')$ (so that $\text{con}(w) = \text{con}(w')$);
(ii) $\text{sim}(w) = \text{sim}(w')$;
(iii) $\text{occ}(x, w) \equiv \text{occ}(x, w') \pmod{2}$ for all $x \in X$.

Further, if $x \in \text{con}(w) = \text{con}(w')$ and $y \in \text{sim}(w) = \text{sim}(w')$, then
Proof. The submonoids \{3,4,5\}, \{1,2,5\}, and \{5,6\} of \(\mathcal{F}\) are isomorphic to \(L_2^1, N_2^1,\) and \(\mathbb{Z}_2\) respectively. Therefore parts (i), (ii), and (iii) follow from Lemma 2.2. Suppose that \(\text{occ}(x, y, w) \neq \text{occ}(x, y, w')\mod 2\) for some \(x \in \text{con}(w) = \text{con}(w')\) and some \(y \in \text{sim}(w) = \text{sim}(w')\), say \(\text{occ}(x, y, w)\) is odd and \(\text{occ}(x, y, w')\) is even. Denote by \(\varphi_1\) the following substitution into the monoid \(\mathcal{F}\):

\[
\varphi_1(z) = 6 \quad \text{if } z = x, \\
\varphi_1(z) = 3 \quad \text{if } z = y, \\
\varphi_1(z) = 5 \quad \text{otherwise.}
\]

Then \(w\varphi_1 = 6 \cdot 3 = 4\) and \(w'\varphi_1 = 5 \cdot 3 = 3\), which is impossible. Hence part (iv) holds.

Suppose that \(x \in \text{con}(w) = \text{con}(w')\) and \(y \in \text{sim}(w) = \text{sim}(w')\) are such that \(x \prec_w y\) and \(x \preceq_{w'} y\). Then it follows from part (i) that within \(w\), every \(x\) occurs before \(y\), while within \(w'\), two occurrences of \(x\) sandwich \(y\). Denote by \(\varphi_2\) the following substitution into \(\mathcal{F}\):

\[
\varphi_2(z) = 3 \quad \text{if } z = x, \\
\varphi_2(z) = 2 \quad \text{if } z = y, \\
\varphi_2(z) = 5 \quad \text{otherwise.}
\]

Then \(w\varphi_2 = 3 \cdot 2 = 2\) and \(w'\varphi_2 = 3 \cdot 2 \cdot 3 = 1\), which is impossible. Therefore part (v) holds. □

12.4. Proof of Proposition 12.1. It is routine to verify that the monoid \(\mathcal{F}\) satisfies the identities (12.1). It remains to show that any identity \(w \approx w'\) satisfied by \(\mathcal{F}\) is implied by the identities (12.1). In view of Lemma 12.2, the words \(w\) and \(w'\) can be assumed to be in canonical form. Therefore by Lemma 12.3(i),

\[
w = \prod_{i=1}^{m}(x_iw_i) \quad \text{and} \quad w' = \prod_{i=1}^{m}(x_iw'_i).
\]

First assume that \((w_1, \ldots, w_{m-1}) = (w'_1, \ldots, w'_{m-1})\). By (II),

\[
w_m = x_1^{e_1} \cdots x_m^{e_m} \quad \text{and} \quad w'_m = x_1^{e'_1} \cdots x_m^{e'_m}
\]

for some \(e_1, e'_1, \ldots, e_m, e'_m \in \{0, 1, 2\}\). Since \(\text{occ}(x_m, w) = 1 + e_m\) and \(\text{occ}(x_m, w') = 1 + e'_m\), it follows from Lemma 12.3(ii)&(iii) that \(e_m = e'_m\). Suppose that \(e_i > e'_i\) for some \(i < m\). Then \(e_i = 2\) and \(e'_i = 0\) by Lemma 12.3(iii). The letter \(x_m\) is then simple in \(w\) by (IV), whence \(x_m\) is also simple in \(w'\) by Lemma 12.3(ii). Then \(x_i \preceq_w x_m\) and \(x_i \preceq_{w'} x_m\), contradicting Lemma 12.3(v). Therefore \(i\) does not exist, whence \(w_m = w'_m\). It follows that the identity \(w \approx w'\) is trivial and so is clearly implied by the identities (12.1).

It remains to assume that \(w_\ell \neq w'_\ell\) for some least possible \(\ell < m\), say

\[
w_\ell = x_1^{e_1} \cdots x_\ell^{e_\ell} \quad \text{and} \quad w'_\ell = x_1^{e'_1} \cdots x_\ell^{e'_\ell}
\]

for some \(e_1, e'_1, \ldots, e_\ell, e'_\ell \in \{0, 1, 2\}\) with \((e_1, \ldots, e_\ell) \neq (e'_1, \ldots, e'_\ell)\). Let

\[
w = p x_\ell w_\ell q \quad \text{and} \quad w' = p x_\ell w'_\ell q'
\]
where \[ p = \prod_{i=1}^{\ell-1} (x_i w_i) = \prod_{i=1}^{\ell-1} (x_i w'_i), \quad q = \prod_{i=\ell+1}^{m} (x_i w_i), \quad \text{and} \quad q' = \prod_{i=\ell+1}^{m} (x_i w'_i). \]

Since \( w_{\ell} \neq w'_{\ell} \), there is some \( k \leq \ell \) such that \( e_k \neq e'_k \). By symmetry, it suffices to assume that \( e_k > e'_k \). Since \( \text{occ}(x_k, x_{\ell+1}, w) = \text{occ}(x_k, p x_{\ell}) + \text{occ}(x_k, w_{\ell}) = \text{occ}(x_k, p x_{\ell}) + e_k, \)

and similarly \( \text{occ}(x_k, x_{\ell+1}, w') = \text{occ}(x_k, p x_{\ell}) + e'_k \), it follows from Lemma 12.3(iv) that \( e_k = 2 \) and \( e'_k = 0 \). Then there are two cases to consider: \( k < \ell \) and \( k = \ell \). It is shown below that these cases lead to contradictions. Therefore \( \ell \) does not exist, whence the identity \( w \approx w' \) is trivial and implied by the identities 12.1.

**Case 1:** \( k < \ell \) (so that \( \ell > 1 \)). Then

\[ w = p x_{\ell} x_{1}^{e_1} \cdots x_{k-1}^{e_{k-1}} x_{k}^{e_k} x_{k+1}^{e_{k+1}} \cdots x_{\ell}^{e_{\ell}} q \]

where \( x_k \notin \text{con}(q) \) by (III) and \( x_{\ell} \) is simple in \( w \) by (IV) (so that \( e_{\ell} = 0 \), and

\[ w' = p x_{\ell} w'_{\ell} q' \]

where \( x_k \notin \text{con}(w'_\ell) \) by assumption and the letter \( x_{\ell} \) is simple in \( w' \) by Lemma 12.3(ii). Since \( x_k \in \text{con}(p) \), the simple letter \( x_{\ell} \) in \( w \) is sandwiched between two occurrences of \( x_k \). By Lemma 12.3(i)&(v), the simple letter \( x_{\ell} \) in \( w' \) is also sandwiched between two occurrences of \( x_k \). Hence \( x_k \in \text{con}(q') \). Further,

\[ \text{occ}(x_k, w) = \text{occ}(x_k, p x_{\ell}) + \text{occ}(x_k, w_{\ell}) + \text{occ}(x_k, q) = \text{occ}(x_k, p x_{\ell}) + 2, \]

\[ \text{occ}(x_k, w') = \text{occ}(x_k, p x_{\ell}) + \text{occ}(x_k, w'_{\ell}) + \text{occ}(x_k, q') = \text{occ}(x_k, p x_{\ell}) + \text{occ}(x_k, q'). \]

Therefore \( \text{occ}(x_k, q') \in \{2, 4, \ldots, \} \) by Lemma 12.3(iii). Since \( q' = \prod_{i=\ell+1}^{m} (x_i w'_i) \), it follows that \( x_k \in \text{con}(w'_r) \) for some \( r \geq \ell + 1 \). There are two subcases.

**Subcase 1.1:** \( \text{occ}(x_k, w'_r) = 2 \) for some \( r \geq \ell + 1 \). Then the letter \( x_r \) is simple in \( w' \) by (IV) and so is sandwiched between two occurrences of \( x_k \). However, \( x_r \) is simple in \( w \) but is not sandwiched between two occurrences of \( x_k \), contradicting Lemma 12.3(v).

**Subcase 1.2:** \( \text{occ}(x_k, w'_r) \leq 1 \) for all \( r \geq \ell + 1 \). Since \( \text{occ}(x_k, q') \in \{2, 4, \ldots, \} \), there exist least possible integers \( p \) and \( q \) such that \( \text{occ}(x_k, w'_p) = \text{occ}(x_k, w'_q) = 1 \) and \( \ell + 1 \leq p < q \). Then

\[ \text{occ}(x_k, x_q, w') = \text{occ}(x_k, p) + \text{occ}(x_k, x_{\ell} w'_\ell) + \text{occ}(x_k, x_{\ell+1} w'_{\ell+1} \cdots x_{p} w'_p) = \text{occ}(x_k, p) + 0 + 1. \]

However, since \( x_k \notin \text{con}(q) \),

\[ \text{occ}(x_k, x_q, w) = \text{occ}(x_k, p) + \text{occ}(x_k, x_{\ell} w_{\ell}) = \text{occ}(x_k, p) + 2. \]

Therefore \( \text{occ}(x_k, x_q, w') \equiv \text{occ}(x_k, x_q, w) \) (mod 2), contradicting Lemma 12.3(iv).

**Case 2:** \( k = \ell \). Then

\[ w = p x_{\ell} x_{1}^{e_1} \cdots x_{\ell-1}^{e_{\ell-1}} x_{\ell}^{e_{\ell}} q \]

where \( x_{\ell} \notin \text{con}(q) \) by (III), and

\[ w' = p x_{\ell} w'_{\ell} q' \]

where \( x_{\ell} \notin \text{con}(w'_\ell) \) by assumption. Since the letter \( x_{\ell} \) is nonsimple in \( w \), it is also nonsimple in \( w' \) by Lemma 12.3(ii), whence \( x_{\ell} \in \text{con}(q') \). Further, since \( \text{occ}(x_{\ell}, w) = 3 \),
it follows from Lemma [12.1](iii) that \( \text{occ}(x_\ell, q') \in \{2, 4, \ldots\} \). Since \( q' = \prod_{i=\ell+1}^m (x_i w_i) \), it follows that \( x_\ell \in \text{con}(w_r) \) for some \( r \geq \ell + 1 \). Contradictions can then be deduced by applying the arguments in Subcases 1.1 and 1.2.

13. The monoid \( G \)

13.1. Main result

Proposition 13.1. The identities
\[
\begin{align*}
x^2yx & \approx yx^3, \quad (13.1a) \\
xyzx & \approx xzxy, \quad (13.1b) \\
xyhtxy & \approx yxhtxy, \quad xhyttxy & \approx xhytyx \quad (13.1c)
\end{align*}
\]
constitute a basis for the monoid \( G \) with the following multiplication table:

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Note that the identity element of the monoid \( G \) is 5.

13.2. A canonical form. In this chapter, a nonsimple word
\[
w = \prod_{i=1}^m (s_i w_i)
\]
is said to be in canonical form if all of the following conditions are satisfied:

(I) the letters of \( s_1 \in X^* \) and \( s_2, \ldots, s_m \in X^+ \) are all simple in \( w \);

(II) the letters of \( w_1, \ldots, w_{m-1} \in X^+ \) and \( w_m \in X^* \) are all nonsimple in \( w \);

(III) for each \( i \), the letters of \( w_i \) are in alphabetical order with \( \text{occ}(x, w_i) \leq 3 \) for any \( x \in X \);

(IV) if \( \text{occ}(x, w_i) = 1 \), then \( \text{occ}(x, w_1), \ldots, \text{occ}(x, w_{i-1}) \leq 1 \);

(V) if \( \text{occ}(x, w_i) = 2 \), then \( \text{occ}(x, w_{i+1}) = \cdots = \text{occ}(x, w_m) = 0 \);

(VI) if \( \text{occ}(x, w_i) = 3 \), then \( \text{occ}(x, w_j) = 0 \) for all \( j \neq i \).

Note that (I) and (II) imply that \( \text{con}(s_i) \cap \text{con}(w_j) = \emptyset \) for any \( i \) and \( j \).

Lemma 13.2. Let \( w \) be any nonsimple word. Then there exists some word \( w' \) in canonical form such that the identities (13.1) imply the identity \( w \approx w' \).

Proof. It suffices to convert the nonsimple word \( w \), using the identities (13.1), into a word in canonical form. By gathering adjacent simple letters and adjacent nonsimple letters in the word \( w \), it is easy to see that \( w \) can be written in the form (13.2) that satisfies (I) and (II). For each \( i \), since the letters in \( w_i \) are nonsimple in \( w \), they can be ordered by the identities (13.1c); then (III) is satisfied by applying the identity \( x^4 \approx x^2 \) from (13.1b). Specifically, each \( w_i \) is of the form \( x_1^{e_1} \cdots x_r^{e_r} \) where \( e_1, \ldots, e_r \in \{1, 2, 3\} \).
If \( \text{occ}(x, w_i) = 1 \) and \( \text{occ}(x, w_j) \in \{2, 3\} \) for some \( j < i \), say \( w_i = w'_i x w''_i \) and \( w_j = w'_j x^{\text{occ}(x, w_j)} w''_j \) where \( w'_i, w''_i, w'_j, w''_j \in (\mathcal{X} \setminus \{x\})^* \), then the identity (13.1a) can be used to gather a factor \( x^2 \) from \( w_j \) with the \( x \) in \( w_i \):

\[
\begin{align*}
    w &= \cdots w'_j x^{\text{occ}(x, w_j)} w''_j \cdots w'_i x w''_i \cdots \overset{13.1a}{\sim} \cdots w'_j x^{\text{occ}(x, w_j) + 2} w''_j \cdots \\
    &\overset{13.1b}{\sim} \left\{ \begin{array}{ll}
        \cdots w'_i w''_i \cdots w'_j x^{3} w''_j \cdots & \text{if } \text{occ}(x, w_j) = 1, \\
        \cdots w'_i w''_i \cdots w'_j x^{\text{occ}(x, w_j)} w''_j \cdots & \text{if } \text{occ}(x, w_j) \in \{2, 3\}.
    \end{array} \right.
\end{align*}
\]

Similarly, if \( \text{occ}(x, w_i) = 2 \) and \( \text{occ}(x, w_j) \in \{1, 2, 3\} \) for some \( j > i \), say \( w_i = w'_i x^2 w''_i \) and \( w_j = w'_j x^{\text{occ}(x, w_j)} w''_j \) where \( w'_i, w''_i, w'_j, w''_j \in (\mathcal{X} \setminus \{x\})^* \), then the identity (13.1a) can be used to gather the factor \( x^2 \) in \( w_i \) with the factor \( x^{\text{occ}(x, w_j)} \) in \( w_j \):

\[
\begin{align*}
    w &= \cdots w'_j x^{\text{occ}(x, w_j)} w''_j \cdots w'_i x w''_i \cdots \overset{13.1a}{\sim} \cdots w'_j x^{\text{occ}(x, w_j)} w''_j \cdots \\
    &\overset{13.1b}{\sim} \left\{ \begin{array}{ll}
        \cdots w'_i w''_i \cdots w'_j x^{3} w''_j \cdots & \text{if } \text{occ}(x, w_j) = 1, \\
        \cdots w'_i w''_i \cdots w'_j x^{\text{occ}(x, w_j)} w''_j \cdots & \text{if } \text{occ}(x, w_j) \in \{2, 3\}.
    \end{array} \right.
\end{align*}
\]

Hence (IV) and (V) are satisfied. If \( \text{occ}(x, w_i) = 3 \) and \( \text{occ}(x, w_j) > 0 \) for some \( j \neq i \), then it is easy to see that the identities (13.1b) can be used to reduce the exponent of \( x \) in \( w_i \) from 3 to 1. Hence (VI) is satisfied.

### 13.3. Identities of the monoid \( \mathcal{G} \)

**Lemma 13.3.** Let \( w \) and \( w' \) be any words in canonical form such that the identity \( w \approx w' \) is satisfied by the monoid \( \mathcal{G} \). Then

(i) \( \text{con}(w) = \text{con}(w') \) and \( \text{sim}(w) = \text{sim}(w') \).

Further, if \( x \in \text{con}(w) = \text{con}(w') \) and \( y, z \in \text{sim}(w) = \text{sim}(w') \), then

(ii) \( \text{occ}(x, w) \equiv \text{occ}(x, w') \pmod{2} \);

(iii) \( \text{occ}(x, y, w) \equiv \text{occ}(x, y, w') \pmod{2} \);

(iv) \( x \ll_w y \) if and only if \( x \ll_{w'} y \);

(v) \( yz \) is a factor of \( w \) if and only if \( yz \) is a factor of \( w' \).

**Proof.** Parts (i) and (ii) follow from Lemma 2.2 since the submonoids \( \{2, 5\} \) and \( \{5, 6\} \) of \( \mathcal{G} \) are isomorphic to \( N_2 \) and \( Z_2 \) respectively.

(iii) Suppose that \( \text{occ}(x, y, w) \not\equiv \text{occ}(x, y, w') \pmod{2} \) for some \( x \in \text{con}(w) = \text{con}(w') \) and \( y \in \text{sim}(w) = \text{sim}(w') \), say \( \text{occ}(x, y, w) = 2p + 1 \) is odd and \( \text{occ}(x, y, w') = 2q \) is even. By part (ii), there exists some \( r \in \{1, 2\} \) such that \( \text{occ}(x, w) + r \equiv 0 \equiv \text{occ}(x, w') + r \pmod{2} \). Denote by \( \varphi_1 \) the following substitution into \( \mathcal{G} \):

\[
    z \mapsto \begin{cases} 
        6 & \text{if } z = x, \\
        2 & \text{if } z = y, \\
        5 & \text{otherwise}.
    \end{cases}
\]

Then

\[
    (x^2w^r)\varphi_1 = 6^{2p+3} \cdot 2 \cdot 6^{\text{occ}(x, w)-(2p+1)+r} = 6 \cdot 6 = 3,
\]

\[
    (x^2w'^r)\varphi_1 = 6^{2q+2} \cdot 2 \cdot 6^{\text{occ}(x, w')-2q+r} = 5 \cdot 2 \cdot 5 = 2,
\]

which is impossible.
13.4. Proof of Proposition 13.1. It is routine to verify that the monoid identities (13.1). It remains to show that any identity \( w \approx x \in G \) the identities (13.1). Since every letter in the factor \( w \) they can be assumed to be in canonical form, say is a simple word. Hence assume that they are both nonsimple. In view of Lemma 13.2, it is easy to show that if \( m_i = 2 \) for all \( i \), then \( \phi \) is contradictorily commutative. Hence every letter in the factor \( u \) is nonsimple in the word \( w' \). Since the word \( w' \) is in canonical form, it follows from (III) that \( u = x_1^{e_1} \cdots x_r^{e_r} \) for some \( x_1, \ldots, x_r \in \sim(w') \) and some \( e_1, \ldots, e_r \in \{1, 2, 3\} \). Since \( \text{occ}(x, z, w) \equiv \text{occ}(x, z, w') \mod 2 \) by part (iii),

\[
\text{occ}(x, y, w) = \text{occ}(x, z, w) \equiv \text{occ}(x, z, w') = \text{occ}(x, y, w') + \text{occ}(x, u) \mod 2.
\]

But \( \text{occ}(x, y, w) \equiv \text{occ}(x, y, w') \mod 2 \) by (iii) so that \( \text{occ}(x, u) \equiv 0 \mod 2 \). Hence \( e_i = 2 \) for all \( i \). It follows from (V) that \( x_i \not\sim w' \) \( z \). Therefore \( x_i \not\sim w \) \( z \) by (iv). It is then obvious that \( x_i \not\sim w \) \( y \), but (iv) is violated since \( x_i \not\sim w' \) \( y \).

13.4. Proof of Proposition 13.1. It is routine to verify that the monoid \( G \) satisfies the identities (13.1). It remains to show that any identity \( w \approx w' \) satisfied by \( G \) is implied by the identities (13.1). Since \( G \) is noncommutative and its submonoid \( \{1, 2, 5\} \) is isomorphic to \( N_2^1 \), it follows from Lemma 13.2 that the identity \( w \approx w' \) is trivial if either \( w \) or \( w' \) is a simple word. Hence assume that they are both nonsimple. In view of Lemma 13.2 they can be assumed to be in canonical form, say

\[
w = \prod_{i=1}^{m} (s_i w_i) \quad \text{and} \quad w' = \prod_{i=1}^{m'} (s'_i w'_i).
\]

It follows from Lemma 13.3(v) that \( m = m' \) and \( s_i = s'_i \) for every \( i \). Hence

\[
w' = \prod_{i=1}^{m} (s_i w'_i).
\]

It is easy to show that if \( m = 1 \), then \( w_1 = w'_1 \) by (II), (III), and Lemma 13.3(i)&(ii), whence the identity \( w \approx w' \) is trivial and so is satisfied by the monoid \( G \). Therefore assume that \( m > 1 \) and let \( \ell \) be the least integer such that \( w_\ell \neq w'_\ell \). Let

\[
w = pw_\ell r \quad \text{and} \quad w' = pw'_\ell r'\]

where \( p = (\prod_{i=1}^{\ell-1} (s_i w_i))s_\ell \), \( r = \prod_{i=\ell+1}^{m} (s_i w_i) \), and \( r' = \prod_{i=\ell+1}^{m} (s_i w'_i) \).
Case 1: $\ell < m$. Then $s_{\ell+1} \neq 0$. Since
\[
\text{occ}(x, \lambda(s_{\ell+1}), w) = \text{occ}(x, p) + \text{occ}(x, w_{\ell}), \\
\text{occ}(x, \lambda(s_{\ell+1}), w') = \text{occ}(x, p) + \text{occ}(x, w'_{\ell}),
\]
Lemma 13.3(iii) implies that $\text{occ}(x, w_{\ell}) \equiv \text{occ}(x, w'_{\ell}) \pmod{2}$ for all $x \in \mathcal{X}$. By symmetry, it suffices to assume that $\text{occ}(x, w_{\ell}) > \text{occ}(x, w'_{\ell})$. Then $(\text{occ}(x, w_{\ell}), \text{occ}(x, w'_{\ell})) \in \{(2, 0), (3, 1)\}$ by (III).

Subcase 1.1: $\text{occ}(x, w_{\ell}) = 2$ and $\text{occ}(x, w'_{\ell}) = 0$. Then it follows from (V) that $\text{occ}(x, r) = 0$ so that $x \preceq \lambda(s_{\ell+1})$. Hence $x \preceq_w \lambda(s_{\ell+1})$ by Lemma 13.3(iv) so that $x \not\in \text{con}(r')$. Now $x \not\in \text{con}(w'_{\ell})$ by assumption and $x \not\in \text{con}(s_{\ell})$ since $x$ is nonsimple in $w'$, and it follows that $x \preceq_w \lambda(s_{\ell})$. However, the assumption $\text{occ}(x, w_{\ell}) = 2$ implies that $x \not\in_w \lambda(s_{\ell})$, whence Lemma 13.3(iv) is violated.

Subcase 1.2: $\text{occ}(x, w_{\ell}) = 3$ and $\text{occ}(x, w'_{\ell}) = 1$. Then it follows from (VI) that $\text{occ}(x, w_{i}) = 0$ for all $i \neq \ell$, whence $\text{occ}(x, p) = \text{occ}(x, r) = 0$ by (I) and (II). Specifically, $x \preceq_w \lambda(s_{\ell+1})$. Now the letter $x$ is nonsimple in $w'$ by Lemma 13.3(i) so that
\[
1 < \text{occ}(x, w') = \text{occ}(x, p) + \text{occ}(x, w'_{\ell}) + \text{occ}(x, r') = 1 + \text{occ}(x, r'),
\]
that is, $x \in \text{con}(r')$. Hence $x \not\in_w \lambda(s_{\ell+1})$ in violation of Lemma 13.3(iv).

Case 2: $\ell = m$ so that $w = pw_m$ and $w' = pw'_m$. Then it follows from Lemma 13.3(ii) that $\text{occ}(x, w_m) \equiv \text{occ}(x, w'_m) \pmod{2}$ for all $x \in \mathcal{X}$. Without loss of generality, assume that $\text{occ}(x, w_m) > \text{occ}(x, w'_m)$. Then $(\text{occ}(x, w_m), \text{occ}(x, w'_m)) \in \{(2, 0), (3, 1)\}$ by (III).

Subcase 2.1: $\text{occ}(x, w_m) = 2$ and $\text{occ}(x, w'_m) = 0$. Since $x \in \text{con}(w) = \text{con}(w')$ by Lemma 13.3(i),
\[
0 < \text{occ}(x, w') = \text{occ}(x, p) + \text{occ}(x, w'_m) = \text{occ}(x, p).
\]
Therefore $x \not\in_w \lambda(s_m)$ and $x \not\in_w \lambda(s_m)$ in violation of Lemma 13.3(iv).

Subcase 2.2: $\text{occ}(x, w_m) = 3$ and $\text{occ}(x, w'_m) = 1$. Then it follows from (VI) that $\text{occ}(x, w_{i}) = 0$ for all $i < m$, whence $\text{occ}(x, p) = 0$ by (I) and (II). Since
\[
\text{occ}(x, w) = \text{occ}(x, p) + \text{occ}(x, w_m) = 3, \\
\text{occ}(x, w') = \text{occ}(x, p) + \text{occ}(x, w'_m) = 1,
\]
it follows that $\text{sim}(w) \neq \text{sim}(w')$, whence Lemma 13.3(i) is violated.

Consequently, Cases 1 and 2 are both impossible, whence the integer $\ell$ does not exist. The identity $w \approx w'$ is trivial and thus satisfied by the monoid $\mathcal{G}$.

14. The monoid $I$

14.1. Main result

Proposition 14.1. The identities
\[
x^2yx \approx xyx^2 \approx yx, \quad x^3 \approx x^2, \quad (14.1a) \\
x^2y^2x \approx x^2y^2 \quad (14.1b)
\]
constitute a basis for the monoid $I$ with the following multiplication table:

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Note that the identity element of the monoid $I$ is 5.

14.2. **A canonical form.** In this chapter, a word $w = w_0 \prod_{i=1}^{m} (x_i w_i)$ (14.2) is said to be in canonical form if $x_1, \ldots, x_m$ are all the simple letters of $w$ and for each $i$, the factor $w_i$ is either empty or of the form $y_1^2 \cdots y_r^2$ for some distinct nonsimple letters $y_1, \ldots, y_r$ of $w$.

**Lemma 14.2.** Let $w$ be any word. Then there exists some word $w'$ in canonical form such that the identities (14.1) imply the identity $w \approx w'$.

**Proof.** It suffices to convert the word $w$, using the identities (14.1), into a word in canonical form. It is easy to show by induction that the identity (14.1b) implies the identity $\alpha_n : x^2 y_1^2 \cdots y_n^2 x^2 \approx x^2 y_1^2 \cdots y_n^2$ for any $n \geq 1$. Let $x_1, \ldots, x_m$ be the simple letters of $w$ in order of first occurrence. Then the word $w$ can be written in the form (14.2) where each factor $w_i$, if nonempty, contains only nonsimple letters of $w$. Apply the identities (14.1a) to replace each letter $y$ in each $w_i$ by $y^2$. Therefore by applying the identity (14.1b) and its consequences $\alpha_n$, each factor $w_i$ is converted to the form $y_1^2 \cdots y_r^2$ where the letters $y_1, \ldots, y_r$ are distinct. ■

14.3. **Proof of Proposition 14.1.** It is routine to verify that the monoid $I$ satisfies the identities (14.1). It remains to show that any identity $w \approx w'$ satisfied by $I$ is implied by the identities (14.1). In view of Lemma 14.2, the words $w$ and $w'$ can be assumed to be in canonical form. Since the submonoid $\{1, 2, 5\}$ of $I$ is isomorphic to $N_1^2$, it follows from Lemma 2.2(ii) that $\text{con}(w) = \text{con}(w')$ and $\text{sim}(w) = \text{sim}(w')$. Further, since $I$ is noncommutative, the order of appearance of the simple letters of $w$ and $w'$ is the same. Therefore

$$w = w_0 \prod_{i=1}^{m} (x_i w_i) \quad \text{and} \quad w' = w'_0 \prod_{i=1}^{m} (x_i w'_i)$$

where $\text{sim}(w) = \text{sim}(w') = \{x_1, \ldots, x_m\}$. Since the submonoid $\{4, 5, 6\}$ of $I$ is isomorphic to $L_2$, it follows from Lemma 2.2(i) that $\text{ini}(w) = \text{ini}(w')$, whence $w_0 = w'_0$. 


Suppose that \( \text{con}(w_m) \neq \text{con}(w'_m) \), say \( y \in \text{con}(w_m) \setminus \text{con}(w'_m) \). Denote by \( \varphi_1 \) the following substitution into the monoid \( I \):

\[
t \mapsto \begin{cases} 
  2 & \text{if } t = x_m, \\
  4 & \text{if } t = y, \\
  5 & \text{otherwise}.
\end{cases}
\]

Then \( w_\varphi_1 \in \{4,5\} \cdot 2 \cdot 4 = \{1\} \) and \( w'_\varphi_1 = 4 \cdot 2 = 2 \), which is impossible. Therefore \( \text{con}(w_m) = \text{con}(w'_m) \). Suppose that \( \text{con}(w_i) \neq \text{con}(w'_i) \) for some \( i \) with \( 0 < i < m \), say \( y \in \text{con}(w_i) \setminus \text{con}(w'_i) \). Denote by \( \varphi_2 \) the following substitution into \( I \):

\[
t \mapsto \begin{cases} 
  2 & \text{if } t = x_i, \\
  6 & \text{if } t = x_{i+1}, \\
  4 & \text{if } t = y, \\
  5 & \text{otherwise}.
\end{cases}
\]

Then \( w_\varphi_2 \in \{4,5\} \cdot 2 \cdot 4 \cdot 6 \cdot \{4,5\} = \{1\} \) and \( w'_\varphi_2 \in \{4,5\} \cdot 2 \cdot 6 \cdot \{4,5\} = \{3\} \), which is impossible. Therefore \( \text{con}(w_i) = \text{con}(w'_i) \) whenever \( 0 < i \leq m \).

Suppose that \( w_i \neq w'_i \) for some \( i \) such that \( 0 < i \leq m \). Then there exist some nonsimple letters \( y \) and \( z \) such that \( w_i = \cdots y^2 \cdots z^2 \cdots \) and \( w'_i = \cdots z^2 \cdots y^2 \cdots \). Denote by \( \varphi_3 \) the following substitution into \( I \):

\[
t \mapsto \begin{cases} 
  2 & \text{if } t = x_i, \\
  4 & \text{if } t = y, \\
  6 & \text{if } t = z, \\
  5 & \text{otherwise}.
\end{cases}
\]

Then \( w_\varphi_3 \in \{4,5,6\} \cdot 2 \cdot 4 \cdot 6 \cdot \{4,5,6\} = \{1\} \) and \( w'_\varphi_3 \in \{4,5,6\} \cdot 2 \cdot 6 \cdot 4 \cdot \{4,5,6\} = \{3\} \), which is impossible. Therefore \( w_i = w'_i \). Consequently, the identity \( w \approx w' \) is trivial and so is implied by the identities (14.1).

### 15. The monoids \( \mathcal{J} \) and \( \mathcal{K} \)

#### 15.1. Main result

**Proposition 15.1.** The identities

\[ x y x^2 \approx x y x, \quad (15.1a) \]
\[ x h y t x y \approx x h y t y x, \quad (15.1b) \]
\[ x^2 y^2 x \approx x^2 y^2 \quad (15.1c) \]

constitute a basis for the monoids \( \mathcal{J} \) and \( \mathcal{K} \) with the following multiplication tables:

\[
\begin{array}{cccccccc}
\mathcal{J} & 1 & 2 & 3 & 4 & 5 & 6 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 & 1 & 2 & 3 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 \\
4 & 1 & 2 & 3 & 4 & 4 & 1 \\
5 & 1 & 2 & 3 & 4 & 5 & 6 \\
6 & 6 & 6 & 6 & 6 & 6 & 6 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
\mathcal{K} & 1 & 2 & 3 & 4 & 5 & 6 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 & 1 & 2 & 3 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 \\
4 & 1 & 2 & 3 & 4 & 4 & 3 \\
5 & 1 & 2 & 3 & 4 & 5 & 6 \\
6 & 6 & 6 & 6 & 6 & 6 & 6 \\
\end{array}
\]
Note that the identity elements of the monoids $\mathcal{J}$ and $\mathcal{K}$ are both 5.

15.2. Identities of the monoids $\mathcal{J}$ and $\mathcal{K}$

**Lemma 15.2.** Let $x, y \in \mathcal{X}$ and $u \in \mathcal{X}^+$ be such that $x, y \notin \text{con}(u)$ and $\text{sim}(u) = \emptyset$.

(i) The identities \[15.1\] imply the identity $u \approx v^2$ for some $v \in \mathcal{X}^+$.

(ii) The identities \[15.1\] imply the identities

\[ xyux^* \approx xyux. \] \[15.2\]

**Proof.** (i) Let $\text{ini}(u) = x_1 \cdots x_m$. By assumption, the letters $x_1, \ldots, x_m$ are all nonsimple in $u$ so that the result clearly holds if $m = 1$. Therefore assume that $m \geq 2$. Write the word $u$ in the form

\[ u = \prod_{i=1}^{m} (x_iu_i) \]

where $u_1, \ldots, u_m$ are possibly empty words such that $\text{con}(u_i) \subseteq \{x_1, \ldots, x_i\}$. Since the letter $x_m$ is nonsimple in the word $u$, all occurrences of $x_m$ in $u$ must belong to the suffix $x_mu_m$, whence $u_m \neq \emptyset$. Let $v = (\prod_{i=1}^{m-1} (x_iu_i))x_m$ so that

\[ u = vu_m. \]

For any word $w$, let $\tilde{w}$ be the word obtained from $w$ by putting its letters in alphabetical order. (For example, if $w = x_7x_2^2x_1x_4x_3^2x_7x_1x_3x_2x_1$ then $\tilde{w} = x_1^3x_2^3x_3^3x_4^2$.) It is clear that $\text{con}(v) = \{x_1, \ldots, x_m\}$, whence $\tilde{v} = x_1^{e_1} \cdots x_m^{e_m}$ for some $e_1, \ldots, e_m \geq 1$.

**Case 1:** $\text{con}(u_m) = \text{con}(v)$, say $\hat{u}_m = x_1^{f_1} \cdots x_{f_m}^{f_m}$ for some $f_1, \ldots, f_m \geq 1$. Since any letter in $u_m$ also occurs in $v$,

\[ u = vu_m \overset{(15.1b)}{=} vx_1^{f_1} \cdots x_{f_m}^{f_m} \overset{(15.1a)}{=} vx_1^{e_1} \cdots x_{e_m}^{e_m} = \tilde{v} \overset{(15.1b)}{=} vv. \]

**Case 2:** $\text{con}(u_m) \neq \text{con}(v)$. Let $x_{\ell_1}$ be the last letter in $v$ that does not occur in $u_m$. Then $v = v'x_{\ell_1}v''$ for some $v', v'' \in \mathcal{X}^*$ such that $x_{\ell_1} \in \text{con}(v') \setminus \text{con}(v'')$ and $\text{con}(v'') \subseteq \text{con}(u_m)$. Suppose that all distinct letters of $v'$, when listed in alphabetical order, are $x_{j_1}, \ldots, x_{j_r}$ (so that $\tilde{v}'' = x_{j_1}^{g_1} \cdots x_{j_r}^{g_r}$ for some $g_1, \ldots, g_r \geq 1$). Then $u_m \overset{=}{=} x_{j_1} \cdots x_{j_r}^r$, for some $u_m \in \mathcal{X}^*$. Since any letter in $u_m$ also occurs in $v$,

\[
uu_m = v'x_{\ell_1}v''u_m \overset{(15.1b)}{=} v'x_{\ell_1}v''x_{j_1} \cdots x_{j_r}^r u_m \overset{(15.1a)}{=} v'x_{\ell_1}^2v''v''u_m \overset{(15.1b)}{=} v'x_{\ell_1}^2v''v''u_m \overset{(15.1a)}{=} v'x_{\ell_1}v''(x_{j_1} \cdots x_{j_r}^r)u_m x_{\ell_1} \overset{(15.1b)}{=} v'x_{\ell_1}v''u_m x_{\ell_1} = uu_m x_{\ell_1}.\]

Therefore $uu_m \overset{(15.1)}{=} uu_m x_{\ell_1}$. If $x_{\ell_2}$ is the last letter in $v$ that does not occur in $u_m x_{\ell_1}$, then repeat the same argument to deduce that $uu_m x_{\ell_1} \overset{(15.1)}{=} uu_m x_{\ell_1} x_{\ell_2}$. It is easy to see how this can be continued so that $uu_m x_{\ell_1} \cdots x_{\ell_{k-1}} x_{\ell_k} \overset{(15.1)}{=} uu_m x_{\ell_1} \cdots x_{\ell_{k-1}} x_{\ell_k}$, where every letter from $v$ belongs to $u_m x_{\ell_1} \cdots x_{\ell_{k-1}} x_{\ell_k}$. It follows that $u \overset{(15.1)}{=} uu_m^*$ with $\text{con}(u_m^*) = \text{con}(v)$. Repeat the argument in Case 1 to deduce that $uu_m^* \overset{(15.1)}{=} vv$. Consequently, $u \overset{(15.1)}{=} vv$. 


(ii) By part (i), there exists some \( v \in \mathcal{X}^+ \) such that \( u \approx v^2 \). Since
\[
\begin{align*}
xyvx & \approx xyv^2x \quad \text{(15.1)} \\
xv^2y & \approx xv^2x \quad \text{(15.1a)} \\
xyv^2 & \approx xyv^2 \quad \text{(15.1c)} \\
xvx & \approx xyxu,
\end{align*}
\]
the identities (15.1) imply the identities (15.2). ■

**Lemma 15.3.** Let \( M \in \{ \mathcal{J}, \mathcal{K} \} \) and let \( w \approx w' \) be any identity satisfied by the monoid \( M \). Then

(i) \( \text{ini}(w) = \text{ini}(w') \) (so that \( \text{con}(w) = \text{con}(w') \));
(ii) \( \text{sim}(w) = \text{sim}(w') \).

Further, if \( x, y \in \text{con}(w) = \text{con}(w') \), then

(iii) \( \text{occ}(x, y, w) = 1 \) if and only if \( \text{occ}(x, y, w') = 1 \).

**Proof.** The submonoids \( \{1, 3, 5\} \) and \( \{1, 2, 5\} \) of \( M \) are isomorphic to \( L_2 \) and \( N_2 \) respectively. Hence parts (i) and (ii) follow from Lemma 2.2. Suppose that \( \text{occ}(x, y, w) = 1 \) and \( \text{occ}(x, y, w') = k \neq 1 \) for some \( x, y \in \text{con}(w) = \text{con}(w') \). If \( k = 0 \), then \( \text{ini}(w) \neq \text{ini}(w') \) and part (i) is violated. Therefore further assume that \( k \geq 2 \). Denote by \( \varphi \) the following substitution into the monoid \( M \):
\[
z \mapsto \begin{cases} 
2 & \text{if } z = x, \\
6 & \text{if } z = y, \\
5 & \text{otherwise}.
\end{cases}
\]
Then \( w\varphi = 2 \cdot 6 \cdots = 3 \) and \( w'\varphi = 2^k \cdot 6 \cdots = 1 \), which is impossible. Hence part (iii) holds. ■

**Lemma 15.4.** Let \( u, u' \in \{x, y, z\}^* \) be such that either \( u = u' = \emptyset \) or \( \lambda(u) = \lambda(u') = z \). Then the monoids \( \mathcal{J} \) and \( \mathcal{K} \) do not satisfy the identity \( x^nyxu \approx x^nyu' \) for any \( n \geq 1 \).

**Proof.** Let \( M \in \{ \mathcal{J}, \mathcal{K} \} \) and let \( \varphi \) be the substitution \( (x, y, z) \mapsto (4, 2, 6) \) into \( M \). Since \( (x^nyzu)\varphi = 1 \cdot (u\varphi) = 1 \) and
\[
(x^nyu')\varphi = 2 \cdot (u'\varphi) = \begin{cases} 
2 & \text{if } u' = \emptyset, \\
3 & \text{otherwise},
\end{cases}
\]
the monoid \( M \) does not satisfy the identity \( x^2yxzu \approx x^2yzu' \). ■

**15.3. A canonical form.** In this chapter, a word
\[
w = x_0^{e_0} \prod_{i=1}^{m} (x_i^{e_i} w_i)
\]
is said to be in **canonical form** if \( x_0, \ldots, x_m \) are distinct letters and \( w_1, \ldots, w_m \) are possibly empty words such that

(I) \( \text{ini}(w) = x_0 \cdots x_m \);
(II) \( e_0, \ldots, e_m \in \{1, 2\} \);
(III) \( w_i \in \{x_0^{f_0} \cdots x_{i-1}^{f_{i-1}} | f_0, \ldots, f_{i-1} \in \{0, 1\}\} \);
(IV) for any \( i \) with \( \text{occ}(x_i, w) \geq 3 \) and any \( j > 1 \), if the \( j \)-th occurrence of \( x_i \) is in \( x_p^{e_r} w_p \) and the \( (j + 1) \)-st occurrence of \( x_i \) is in \( x_r^{e_r} w_r \) for some \( r > p \), then there exists \( q \) with \( p < q \leq r \) such that \( e_q = 1 \) and \( x_q \notin \text{con}(w_{q+1} \cdots w_r) \).
Note that since $x_i \notin \text{con}(w_1 \cdots w_i)$ by (III), it follows from (IV) that

(V) $x_q \notin \text{con}(w_1 \cdots w_r)$ in (IV).

**Lemma 15.5.** Let $w$ be any word. Then there exists some word $w'$ in canonical form such that the identities \((15.1)\) imply the identity $w \approx w'$.

**Proof.** It suffices to convert the word $w$, using the identities \((15.1)\), into a word in canonical form. Without loss of generality, assume that $\text{ini}(w) = x_0 \cdots x_m$. Then $w$ can be written in the form

$$w = x_0^m \prod_{i=1}^{m} (x_i^{e_i}w_i)$$

for some $e_0, \ldots, e_m \geq 1$ and some $w_1, \ldots, w_m \in \mathcal{X}^*$ such that $\text{con}(w_i) \subseteq \{x_1, \ldots, x_i\}$ for all $i$. For each $i$, the letters in $w_i$ are not first occurrences in $w$ so that the identities \((15.1b)\) can be used to permute them in any manner within $w_i$. Specifically, any $x_i$ in the factor $w_i$ can be moved to the left and gathered with the factor $x_i^{e_i}$ that precedes $w_i$, and the rest of the letters in $w_i$ can be ordered alphabetically. Hence for each $i$, it can be assumed that $w_i = x_0^{j_0} \cdots x_i^{j_i-1}$. It is then easy to see that (II) and (III) are satisfied by applying the identities \((15.1a)\).

It remains to show that (IV) can be satisfied. Let $\text{occ}(x_i, w) \geq 3$ and $j > 1$. Suppose that the $j$th occurrence of $x_i$ is in $x_p^{e_p}w_p$ and the $(j + 1)$st occurrence of $x_i$ is in $x_r^{e_r}w_r$ for some $r > p$. Then it suffices to achieve either one of the following:

(a) $e_q = 1$ and $x_q \notin \text{con}(w_{q+1} \cdots w_r)$ for some $p < q \leq r$;

(b) eliminate the $(j + 1)$st occurrence of $x_i$ from $w$ by the identities \((15.1)\).

First observe that

- $x_i$ cannot be $x_r$, since $i = r > p$ implies that $x_i$ is, by (III), contradictorily not in $x_p^{e_p}w_p$;
- if $x_i = x_p$ (that is, the factor $x_p^{e_p}w_p$ contains the first and $j$th occurrences of $x_i$ with $j > 1$), then it follows from (III) that $e_p = 2$, $j = 2$, and the factor $x_p^{e_p}$ consists of the first and second $x_i$ of $w$.

Therefore $x_p^{e_p}w_p = w'_p x_i w''_p$ and $w_r = w'_r x_i w''_r$ for some $w'_p, w''_p, w'_r, w''_r \in \mathcal{X}^*$. Denote by $\varphi$ the substitution $x \mapsto x^2$ for all $x \in \mathcal{X}$. Since the letters in $w_1, \ldots, w_m$ are all nonsimple in $w$,

$$w = x^{e_p}w_p \cdots (w'_p x_i w''_p)(x^{e_p+1}w_{p+1}) \cdots (x^{e_r}w_r x_i w''_r) \cdots$$

\((15.1)\)

$$\approx \cdots (w'_p x_i w''_p)(x^{e_p+1}w_{p+1}) \cdots (x^{e_r}w_r x_i w''_r) \cdots$$

\((15.1a)\)

$$\approx \cdots w'_{p+1} x_i (w''_{p+1} \varphi)(x^{e_p+1}(w_{p+1} \varphi)) \cdots (x^{e_r}w'_r w''_r x_i) \cdots$$

**Case 1:** $\text{sim}(z) \neq \emptyset$. Since the letters in $w''_p \varphi, w_{p+1} \varphi, \ldots, w_{r-1} \varphi, w'_r \varphi$, and $w''_r \varphi$ are all nonsimple in $z$, at least one of the letters $x_{p+1}, \ldots, x_r$ is simple in $z$. Therefore there exists $q$ with $p < q \leq r$ such that $e_q = 1$ and $x_q \notin \text{con}(w_{q+1} \cdots w_r)$, whence (a) holds.
Case 2: $\text{sim}(z) = \emptyset$. Since the $x_i$ that immediately precedes $z$ is the $j$th occurrence in $w$ with $j > 1$, the factor $z$ is preceded by two or more occurrences of $x_i$. It then follows from Lemma 15.2 that

$$w \overset{15.1}{=} \cdots w_i' x_i z x_i \cdots \overset{15.2}{=} \cdots w_i^r x_i \cdots$$

$$\overset{15.1a}{=} \cdots (w_i^p x_i w_i') (x_i^{e_{p+1}} w_{p+1}) \cdots (x_i^{e_{r-1}} w_{r-1}) (x_i^{e_r} w_i^r w_i') \cdots = \cdots (x_i^{e_p} w_p) (x_i^{e_{p+1}} w_{p+1}) \cdots (x_i^{e_{r-1}} w_{r-1}) (x_i^{e_r} w_i^r w_i') \cdots .$$

Hence the $(j + 1)$st occurrence of $x_i$ in $w$ is eliminated and (b) holds. □

15.4. Proof of Proposition 15.1. Let $M \in \{J, K\}$. It is routine to verify that the monoid $M$ satisfies the identities (15.1). It remains to show that any identity $w \cong w'$ satisfied by $M$ is implied by the identities (15.1). In view of Lemma 15.5, the words $w$ and $w'$ can be assumed to be in canonical form. Therefore by Lemma 15.3(i),

$$w = x_0^{e_0} \prod_{i=1}^m (x_i^{e_i} w_i) \quad \text{and} \quad w' = x_0^{e_0'} \prod_{i=1}^m (x_i^{e_i'} w_i').$$

It follows from (II) and (III) that $\text{occ}(x_m, w) = e_m$ and $\text{occ}(x_m, w') = e_m'$ are in \{1, 2\}, whence $e_m = e_m'$ by Lemma 15.3(ii). If $i < m$, then $\text{occ}(x_i, x_{i+1}, w) = e_i$ and $\text{occ}(x_i, x_{i+1}, w') = e_i'$ are in \{1, 2\}, whence $e_i = e_i'$ by Lemma 15.3(iii). Hence $(e_0, \ldots, e_m) = (e_0', \ldots, e_m')$. Working toward a contradiction, suppose that there exists a least integer $r$ such that $w_r \neq w'_r$. Then

$$w = hx_r^{e_r} w_r t \quad \text{and} \quad w' = hx_r^{e_r'} w'_r t'$$

with $h = x_0^{e_0} \prod_{i=1}^{r-1} (x_i^{e_i} w_i)$, $t = \prod_{i=r+1}^m (x_i^{e_i} w_i)$, and $t' = \prod_{i=r+1}^m (x_i^{e_i'} w_i')$, where

(a) $t = t' = \emptyset$ if $r = m$ (and $\lambda(t) = \lambda(t') = x_{r+1}$ if $r < m$).

Since $w_r \neq w'_r$, there exists some $\ell$ such that $x_{\ell}$ belongs to either $w_r$ or $w'_r$ but not both. By symmetry, it suffices to assume that $x_{\ell} \in \text{con}(w_r) \setminus \text{con}(w'_r)$. It then follows from (III) that

(b) $\ell < r$, $\text{occ}(x_{\ell}, w_r) = 1$, and $\text{occ}(x_{\ell}, w'_r) = 0$.

Since every letter of $w_r$ is nonsimple in $w$, the factor $h$ contains some $x_{\ell}$, say the last $x_{\ell}$ in $h$ occurs in the factor $x_\ell^{e_\ell} w_p$. Then by (IV), there exists some $q$ with $p < q \leq r$ such that $e_q = 1$ and $x_q \notin \text{con}(w_{q+1} \cdots w_r)$. Further, since $x_q \notin \text{con}(w_1 \cdots w_r)$ by (V),

(c) $x_{\ell} \ll_h x_q$, $\text{occ}(x_q, h) = 1$, and $\text{occ}(x_q, w_r) = 0$.

Suppose that $x_q \in \text{con}(w'_r)$. Then

$$\text{occ}(x_q, w') \geq \text{occ}(x_q, h) + \text{occ}(x_q, w'_r) = 1 + 1 = 2$$

by (c) so that $\text{occ}(x_q, w) \geq 2$ by Lemma 15.3(ii). Since

$$2 \leq \text{occ}(x_q, w) = \text{occ}(x_q, h) + \text{occ}(x_q, w_r) + \text{occ}(x_q, t) = 1 + 0 + \text{occ}(x_q, t)$$
by (c), it follows that $\text{occ}(x_q, t) \geq 1$. Hence $t$ is nonempty with $\lambda(t) = \lambda(t') = x_{r+1}$. It follows from (c) that
\[
\text{occ}(x_q, x_{r+1}, w) = \text{occ}(x_q, h) + \text{occ}(x_q, w_r) = 1 + 0 = 1,
\]
\[
\text{occ}(x_q, x_{r+1}, w') = \text{occ}(x_q, h) + \text{occ}(x_q, w'_r) = 1 + 1 = 2,
\]
contradicting Lemma [15.3](iii). Thus the condition $x_q \in \text{con}(w'_r)$ is impossible, whence (d) $\text{occ}(x_q, w'_r) = 0$.

Let $\varphi$ be the substitution $x \mapsto 1$ for all $x \in \mathcal{X} \setminus \{x_\ell, x_q, x_{r+1}\}$ and let $n = \text{occ}(x_\ell, h)$. Then $h\varphi = x_\ell^n x_q$ by (c), $w_r \varphi = x_\ell$ by (b) and (c), and $w'_r \varphi = 1$ by (b) and (d). Therefore
\[
\begin{align*}
\text{w} \varphi &= (h \varphi)(x_\ell^n \varphi)(w_r \varphi)(t \varphi) = x_\ell^n x_q \cdot 1 \cdot x_\ell \cdot (t \varphi) = x_\ell^n x_q x_\ell (t \varphi), \\
\text{w'} \varphi &= (h \varphi)(x_\ell^n \varphi)(w'_r \varphi)(t' \varphi) = x_\ell^n x_q \cdot 1 \cdot 1 \cdot (t' \varphi) = x_\ell^n x_q (t' \varphi),
\end{align*}
\]
where $t \varphi, t' \varphi \in \{x_\ell, x_q, x_{r+1}\}^*$. Since the identity $x_\ell^n x_q x_\ell (t \varphi) \approx x_\ell^n x_q (t' \varphi)$ is obtained from $w \approx w'$ by eliminating all occurrences of some letters, it is satisfied by $M$. However, it follows from (a) that either $t \varphi = t' \varphi = \emptyset$ or $\lambda(t \varphi) = \lambda(t' \varphi) = x_{r+1}$, which is impossible in view of Lemma [15.4](iv).

A. Multiplication tables of monoids of order six

The multiplication tables of all 1373 distinct monoids of order six are lexicographically listed in rows (A.0)–(A.137) below. The underlying set of each monoid is $\{1, 2, 3, 4, 5, 6\}$, and each table is given by a $6 \times 6$ array where the $(i, j)$-entry denotes the product of the elements $i$ and $j$. Each monoid is identified in one of the following ways:

**Commutativity or idempotency**

- The table of a monoid that is commutative or idempotent is labeled with (Com) or (Idem) respectively.
- The table of a semilattice is labeled with (S.L.). Recall that a semilattice is a semigroup that is both commutative and idempotent.
- The table of the commutative (cyclic) group of order six is labeled with $\mathbb{Z}_6$.

**Conditions**

- If a noncommutative, nonidempotent monoid is finitely based by Condition $m$ or its dual condition, where $m \in \{1, 2, \ldots, 9\}$, then its table is labeled with $(m)$.
- The symmetric group on three letters is finitely based by Condition 2 and its table is labeled with $S_3$.

---

A commutative monoid is finitely based by Condition 1 and a finite idempotent monoid is finitely based by Condition 2. However, it has been known since the late 1960s that a semigroup is finitely based if it is either commutative or idempotent.

Although the groups $S_3$ and $\mathbb{Z}_6$ are finitely based by Condition 2, the finite basis property of every finite group was already established by Oates and Powell in the early 1960s.
### Sporadic cases

- The table of each of the 13 sporadic monoids in Chapter 3 is labeled with the monoid's symbol.

The following records the rows in which the tables of the 13 sporadic monoids are located:

<table>
<thead>
<tr>
<th>Monoid</th>
<th>$A$</th>
<th>$B$</th>
<th>$C$</th>
<th>$D$</th>
<th>$E$</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>Monoid</th>
<th>$F$, $G$, $H$, $I$, $J$, $K$, $B_2^1$, $A_2^1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row</td>
<td>(A.57)</td>
</tr>
</tbody>
</table>

The tables of the groups $Z_6$ and $S_3$ are located in row (A.137).
Minimal non-finitely based monoids

(A.7)

(A.8)

(A.9)

(A.10)

(A.11)

(A.12)

(A.13)

(A.14)

(A.15)
Minimal non-finitely based monoids

(A.43)

(A.44)

(A.45)

(A.46)

(A.47)

(A.48)

(A.49)

(A.50)

(A.51)
Minimal non-finitely based monoids

(A.79)

(A.80)

(A.81)

(A.82)

(A.83)

(A.84)

(A.85)

(A.86)

(A.87)
Minimal non-finitely based monoids

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