VARIETIES GENERATED BY 2-TESTABLE MONOIDS

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Abstract

The smallest monoid containing a 2-testable semigroup is defined to be a 2-testable monoid. The well-known Brandt monoid $B_2^1$ of order six is an example of a 2-testable monoid. The finite basis problem for 2-testable monoids was recently addressed and solved. The present article continues with the investigation by describing all monoid varieties generated by 2-testable monoids. It is shown that there are 28 such varieties, all of which are finitely generated and precisely 19 of which are finitely based. As a comparison, the subvariety lattice of the monoid variety generated by the monoid $B_2^1$ is examined. This lattice has infinite width, satisfies neither the ascending chain condition nor the descending chain condition, and contains non-finitely generated varieties.

1. Introduction

A class of algebras is a variety if it is closed under the formation of homomorphic images, subalgebras, and arbitrary direct products. An algebra is finitely based if the identities it satisfies are finitely axiomatizable. The algebras considered in the present article are semigroups and monoids. For any class $\mathcal{C}$ of semigroups or monoids, let $V_S \mathcal{C}$ denote the semigroup variety generated by $\mathcal{C}$. For any class $\mathcal{C}$ of monoids, let $V_M \mathcal{C}$ denote the monoid variety generated by $\mathcal{C}$. Refer to the surveys of Shevrin and Volkov [20] and Volkov [28] for a wealth of information on varieties, identities, and the finite basis problem for semigroups and monoids.

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A semigroup is 2-testable if it satisfies any identity formed by a pair of words that begin with the same letter, end with the same letter, and share the same set of factors of length two. Trahtman proved that the class of all 2-testable semigroups coincides with the variety $A_2 = \mathcal{V}_S \{A_2\}$ generated by the 0-simple semigroup

$$A_2 = \langle a, b \mid a^2 = aba = a, \ b^2 = 0, \ bab = b \rangle$$

of order five [26], and that the identities

$$(1.1) \quad x^3 \approx x^2, \ xyxy \approx xy, \ yxyx \approx zxyz$$

constitute a finite basis for the variety $A_2$ [25]. It follows that any semigroup that satisfies the identities (1.1) is 2-testable. In particular, the Brandt semigroup

$$B_2 = \langle a, b \mid a^2 = b^2 = 0, \ aba = a, \ bab = b \rangle$$

of order five is 2-testable; this semigroup was also shown by Trahtman to be finitely based [22].

For any semigroup $S$, let $S^1$ denote the smallest monoid containing $S$. Since the variety $A_2$ coincides with the class of 2-testable semigroups [26], it is convenient to refer to a monoid of the form $S^1$, where $S$ is any semigroup from $A_2$, as a 2-testable monoid. The 2-testable monoids $A_1^1$ [23] and $B_2^1$ [17] are non-finitely based. In fact, a well-known result of M. V. Sapir [19] implies that the monoids $A_1^1$ and $B_2^1$ are inherently non-finitely based, that is, they are not contained in any finitely based locally finite variety. It follows that any 2-testable monoid $S^1$ is non-finitely based whenever $B_2 \notin \mathcal{V}_S \{S\}$. Recently, the finite basis property of all 2-testable monoids $S^1$ for which $B_2 \notin \mathcal{V}_S \{S\}$ was established [14]. These results led to a solution of the finite basis problem for all 2-testable monoids.

**Theorem 1.1.** For any semigroup $S \in A_2$, the monoid $S^1$ is finitely based if and only if $B_2 \notin \mathcal{V}_S \{S\}$.

The present article continues with the investigation of 2-testable monoids by describing the monoid varieties they generate. Since the variety $A_2$ is the largest variety generated by 2-testable semigroups [26], it follows that the lattice $\mathcal{L}(A_2)$ of subvarieties of $A_2$ coincides with the lattice of all varieties of 2-testable semigroups. This lattice is countably infinite [10] and contains an isomorphic copy of any finite lattice [27]. In contrast, there are only 28 monoid varieties generated by 2-testable monoids; these varieties constitute the join-semilattice in Figure 1.

In Section 3, finite 2-testable semigroups are presented to show that the varieties in Figure 1 are all finitely generated by 2-testable monoids. Other information on these varieties that are required in later sections are also given. In particular, it is noted that the monoid variety $A_0^1 \lor L_2^1 \lor R_2^1$ coincides with the largest finitely based variety generated by 2-testable monoids.
All subvarieties of $A_1^1 \lor L_2^1 \lor R_2^1$ are then identified in Section 4. These subvarieties coincide with the 19 finitely based varieties in Figure 1 and constitute a sublattice of the lattice of monoid varieties.

In Section 5, non-finitely based varieties generated by 2-testable monoids are examined. It is easily seen from Theorem 1.1 that these varieties belong to the interval $[B_2^1, A_2^1]$, where $A_2^1 = \mathbb{V}_M \{A_2^1\}$ and $B_2^1 = \mathbb{V}_M \{B_2^1\}$. Since all varieties in the interval $[B_2^1, A_2^1]$ are non-finitely based, none of them has a sufficiently well-described identity basis. It is thus extremely difficult, if not impossible, to identify all varieties in the interval $[B_2^1, A_2^1]$. Fortunately, based on recent results regarding subvarieties of $A_2^1$ [13], the varieties in the interval $[B_2^1, A_2^1]$ generated by 2-testable monoids are shown to coincide with the nine non-finitely based varieties in Figure 1. These nine varieties constitute a join-semilattice, but unlike the 19 finitely based varieties in Section 4, it is unknown if this join-semilattice is a lattice. Nevertheless, the join-semilattice in Figure 1 is verified by results established in Sections 3–5.
Now regardless of whether or not the join-semilattice in Figure 1 is a lattice, the varieties it contains are far from all subvarieties of $A^2_1$. This is demonstrated in Section 6, where results of Jackson [5], Jackson and O. Sapir [7], and M. V. Sapir [18, 19] are used to establish extreme properties of the lattice of subvarieties of the smaller variety $B^2_1$. Specifically, this lattice has infinite width, satisfies neither the ascending chain condition nor the descending chain condition, and contains non-finitely generated varieties.

The article ends with Section 7.0, where several open questions regarding the join-semilattice in Figure 1 and subvarieties of $A^2_1$ are posed.

2. Preliminaries

Most of the notation and background material of this article are given in this section. Refer to the monograph of Burris and Sankappanavar [2] for more information on universal algebra.

2.1. Letters and words

Let $\mathcal{X}$ be a fixed countably infinite alphabet throughout. Denote by $\mathcal{X}^+$ and $\mathcal{X}^*$ the free semigroup and the free monoid over $\mathcal{X}$ respectively. Elements of $\mathcal{X}$ and $\mathcal{X}^*$ are referred to as letters and words respectively.

Let $x$ be any letter and $w$ be any word. Then

- the content of $w$, denoted by $\text{con}(w)$, is the set of letters occurring in $w$;
- the head of $w$, denoted by $\text{h}(w)$, is the first letter occurring in $w$;
- the tail of $w$, denoted by $t(w)$, is the last letter occurring in $w$;
- the initial part of $w$, denoted by $\text{ini}(w)$, is the word obtained from $w$ by retaining the first occurrence of each letter;
- the final part of $w$, denoted by $\text{fin}(w)$, is the word obtained from $w$ by retaining the last occurrence of each letter;
- the number of times $x$ occurs in $w$ is denoted by $\text{occ}(x, w)$;
- $x$ is simple in $w$ if $\text{occ}(x, w) = 1$;
- $w$ is simple if $\text{occ}(y, w) \leq 1$ for any $y \in \mathcal{X}$;
- $w$ is quadratic if $\text{occ}(y, w) \leq 2$ for any $y \in \mathcal{X}$.

Let $w$ be any quadratic word. If $w = axbxc$ for some $x \in \mathcal{X}$ and $a, b, c \in \mathcal{X}^*$ such that $x \notin \text{con}(abc)$, then the distance between the two occurrences of $x$ in $w$ is the length of $b$. If $x_1, \ldots, x_r$ are all the distinct non-simple
letters of \( w \), then the *separation degree* of \( w \) is the sum \( d_1 + \cdots + d_r \) where \( d_i \) is the distance between the two occurrences of \( x_i \) in \( w \).

### 2.2. Identities and varieties

An identity is written in the form \( w \approx w' \) where \( w, w' \in \mathcal{X}^+ \). An identity \( w \approx w' \) is *nontrivial* if \( w \neq w' \). A semigroup \( S \) *satisfies* an identity \( w \approx w' \) if, for any substitution \( \varphi \) from \( \mathcal{X} \) into \( S \), the elements \( w\varphi \) and \( w'\varphi \) coincide in \( S \). A class \( \mathcal{C} \) of semigroups *satisfies* an identity \( w \approx w' \) if every semigroup in \( \mathcal{C} \) satisfies \( w \approx w' \); this is indicated by \( \mathcal{C} \models w \approx w' \).

Let \( \Sigma \) be any set of identities. The deducibility of an identity \( w \approx w' \) from \( \Sigma \) is indicated by \( w \Sigma w' \). The monoid variety *defined* by \( \Sigma \), denoted by \([\Sigma]\), is the class of all monoids that satisfy all identities in \( \Sigma \); in this case, \( \Sigma \) is said to be a *basis* for the variety. A variety is *finitely based* if it possesses a finite basis.

For any variety \( \mathcal{V} \) and any subvariety \( \mathcal{V}' \) of \( \mathcal{V} \), the *interval* \([\mathcal{V}', \mathcal{V}]\) is the set of all subvarieties of \( \mathcal{V} \) containing \( \mathcal{V}' \). Let \( \mathcal{L}(\mathcal{V}) \) denote the lattice of subvarieties of \( \mathcal{V} \). Equivalently, \( \mathcal{L}(\mathcal{V}) = [\mathcal{0}, \mathcal{V}] \) where \( \mathcal{0} \) is the trivial variety.

**Lemma 2.1** (Almeida [1, Lemma 7.1.1]). *Let \( S \) be any semigroup and \( \mathcal{V} \) be any monoid variety such that \( S \in \mathcal{V}_S \mathcal{V} \). Then \( S^1 \in \mathcal{V} \).*

### 3. Some 2-testable monoids

The present section introduces 2-testable monoids that generate the varieties in Figure 1. Some identities and identity bases that are required in later sections are also given.

#### 3.1. Monoids generating varieties in Figure 1

It is routinely checked that the following semigroups satisfy the identities (1.1) and so are 2-testable:

\[
A_0 = \langle a, b \mid a^2 = a, \ b^2 = b, \ ba = 0 \rangle,
\]

\[
B_0 = \langle a, b, c \mid a^2 = a, \ b^2 = b, \ ab = ba = 0, \ ac = cb = c \rangle,
\]

\[
I = \langle a, b \mid ab = a, \ ba = 0, \ b^2 = b \rangle,
\]

\[
J = \langle a, b \mid ba = a, \ ab = 0, \ b^2 = b \rangle,
\]

\[
K = \langle a, b \mid a^2 = b^2 = ba = 0 \rangle.
\]
\[ L_2 = \langle a, b \mid a^2 = ab = a, \ b^2 = ba = b \rangle, \]
\[ N_2 = \langle a \mid a^2 = 0 \rangle, \]
\[ R_2 = \langle a, b \mid a^2 = ba = a, \ b^2 = ab = b \rangle. \]

In fact, the semigroups \( A_0, I, J, L_2, N_2, \) and \( R_2 \) are isomorphic to subsemigroups of \( A_2 \), and the semigroup \( B_0 \) is isomorphic to a subsemigroup of \( B_2 \). Note that \( L_2 \) is a left-zero semigroup, \( N_2 \) is a null semigroup, and \( R_2 \) is a right-zero semigroup. Let \( Y \) be the variety of semilattice monoids and let

\[ A_1^0 = \mathbb{V}_M \{ A_0^1 \}, \quad B_0^1 = \mathbb{V}_M \{ B_0^1 \}, \quad I^1 = \mathbb{V}_M \{ I^1 \}, \quad J^1 = \mathbb{V}_M \{ J^1 \}, \]
\[ K^1 = \mathbb{V}_M \{ K^1 \}, \quad L_2^1 = \mathbb{V}_M \{ L_2^1 \}, \quad N_2^1 = \mathbb{V}_M \{ N_2^1 \}, \quad R_2^1 = \mathbb{V}_M \{ R_2^1 \}. \]

The join of two varieties \( \mathcal{V} \) and \( \mathcal{V}' \), denoted by \( \mathcal{V} \vee \mathcal{V}' \), is the smallest variety containing \( \mathcal{V} \) and \( \mathcal{V}' \). If \( S \) and \( T \) are 2-testable semigroups, then the direct product \( S \times T \) is a 2-testable semigroup such that \( \mathbb{V}_M \{ S^1 \} \vee \mathbb{V}_M \{ T^1 \} = \mathbb{V}_M \{ (S \times T)^1 \} \). It follows that the varieties in Figure 1 are finitely generated by 2-testable monoids.

**Lemma 3.1.** The variety \( A_1^0 \vee L_2^1 \vee R_2^1 \) is the largest finitely based variety generated by 2-testable monoids.

**Proof.** The variety \( A_2 \) contains a subvariety that is largest with respect to not containing the semigroup \( B_2 \) [8, Theorem 3.6]; this subvariety of \( A_2 \) is generated by a certain 2-testable semigroup \( C_0 \) of order six [15, Theorem 4.2(iii)]. By Theorem 1.1, the variety \( \mathbb{V}_M \{ C_0^1 \} \) coincides with the largest finitely based variety generated by 2-testable monoids. The present lemma then follows since the varieties \( \mathbb{V}_M \{ C_0^1 \} \) and \( \mathbb{V}_M \{ A_0^1 \times L_2^1 \times R_2^1 \} \) coincide [14, Lemma 3.2].

### 3.2. Bases and identities

**Lemma 3.2.** Let \( w \approx w' \) be any identity. Then

(i) \( L_2^1 \models w \approx w' \) if and only if \( \text{ini}(w) = \text{ini}(w') \);

(ii) \( R_2^1 \models w \approx w' \) if and only if \( \text{fin}(w) = \text{fin}(w') \).

Further, if the words \( w \) and \( w' \) are quadratic, then

(iii) \( N_2^1 \models w \approx w' \) if and only if \( \text{occ}(x, w) = \text{occ}(x, w') \) for all \( x \in X \).

**Proof.** These results are well known and easily verified. \( \square \)
Lemma 3.3.
(i) $L_1^2 = [xyx \approx xy]$.
(ii) $R_1^1 = [xyx \approx yx]$.
(iii) $L_1^2 \lor R_1^1 = [x^2 \approx x, xyxxz \approx xyxz]$.
(iv) $N_1^2 \lor L_1^2 = [x^3 \approx x^2, xy \approx x^2y]$.
(v) $N_1^2 \lor R_1^1 = [x^3 \approx x^2, xy \approx yx^2]$.

Proof. Parts (i)–(iii) are well known and can be found in Almeida [1, Section 5.5]. The arguments of Edmunds [3, proof of Proposition 3.1(c)] can be repeated to establish part (iv). Part (v) is symmetrical to part (iv).

The following identities are required in the bases for some varieties containing the monoids $A_0^1$ and $B_0^1$:

\[
\begin{align*}
\text{(⋆)} & \quad xyxz \approx xyzx, \\
\text{(♦)} & \quad xyxy \approx x^2y^2, \\
\text{(▷)} & \quad xyxy \approx x^2x, \\
\text{(◆)} & \quad xyxy \approx yx^2y.
\end{align*}
\]

Lemma 3.4.
(i) $A_0^1 \lor L_1^2 \lor R_1^1 = [(\text{⋆})]$.
(ii) $A_0^1 \lor L_1^1 = [(\text{⋆}), (\text{▷})]$.
(iii) $A_0^1 \lor R_1^1 = [(\text{⋆}), (\text{◆})]$.
(iv) $A_0^1 = [(\text{⋆}), (\text{▷}), (\text{◆})]$.
(v) $B_0^1 \lor L_1^2 \lor R_1^1 = [(\text{⋆}), (\text{♦})]$.
(vi) $B_0^1 \lor L_1^1 = [(\text{⋆}), (\text{♦}), (\text{▷})]$.
(vii) $B_0^1 \lor R_1^1 = [(\text{⋆}), (\text{♦}), (\text{◆})]$.
(viii) $B_0^1 = [(\text{⋆}), (\text{♦}), (\text{▷}), (\text{◆})]$.

Proof. Parts (i)–(iv) follow from Lee [14, Proposition 2.3], part (viii) follows from Edmunds [3, Proposition 3.1(i)], and part (v) is established at the end of this subsection.

It follows from Edmunds [3, Proposition 3.1(i)] that the identities \{ (\text{⋆}), (\text{♦}), (\text{▷}) \} constitute a basis for a certain monoid of order five. The proof of
this result can easily be repeated to show the same identities also constitute a basis for the variety $B_1^1 \lor L_2^1$. Hence part (vi) holds. By symmetry, part (vii) also holds. □

**Remark 3.5.** Note that if a letter $x$ occurs three or more times in a word $w$, then all except the first and last occurrences of $x$ in $w$ can be eliminated by the identity $(\star)$ and its consequences $x^2yx \approx xyx \approx xyx^2$. Therefore any word can be converted by the identity $(\star)$ into a unique quadratic word.

Let $x$ be any letter in a quadratic word $w$ such that $w = axbxc$ for some $a, b, c \in X^*$. Then $x$ is said to be **tight** in $w$ if $h(b) \notin \text{con}(a)$ and $t(b) \notin \text{con}(c)$. Note that $x$ is vacuously tight in $w$ if $b = \emptyset$. A quadratic word is **tight** if all its non-simple letters are tight in it.

**Lemma 3.6.** Let $w$ be any word. Then there exists a tight quadratic word $\tilde{w}$ such that the identity $w \approx \tilde{w}$ is a consequence of the identities $\{(\star), (\bullet)\}$.

**Proof.** By Remark 3.5, the word $w$ can be chosen to be quadratic. Suppose that $w = axbxc$ for some $a, b, c \in X^*$ and $x$ is not tight in $w$. If $h(b) = h \in \text{con}(a)$, say $a = a'ha''$ and $b = h'b'$ for some $a', a'', b' \in X^*$, then the identities $\{(\star), (\bullet)\}$ can be applied to interchange the first $x$ with the first letter of $b$:

$$w = a'ha''xhx'h'bx'c \approx a'ha''hxhx'b'xc \approx a'ha''h^2x^2b'xc \approx ahxb'xc.$$

Similarly, if $t(b) \in \text{con}(c)$, then the identities $\{(\star), (\bullet)\}$ can be applied to interchange the second $x$ with the last letter of $b$. These interchanges can be repeated until the letter $x$ is tight. In other words, the identities $\{(\star), (\bullet)\}$ **tightened** the letter $x$. Observe that in the process of tightening a non-simple letter $x$, each time an identity from $\{(\star), (\bullet)\}$ is applied to interchange $x$ with another non-simple letter $y$,

- the distance between the two occurrences of $x$ decreases,
- the distance between the two occurrences of $y$ decreases, and
- the distance between the two occurrences of any other non-simple letter remains unchanged.

It follows that whenever the identities $\{(\star), (\bullet)\}$ tighten a non-tight letter $x$, the separation degree of the word decreases. Since the separation degree of any word is nonnegative, only finitely many applications of the identities $\{(\star), (\bullet)\}$ are required to tighten every non-simple letter of $w$. □

**Proof of Lemma 3.4(v).** Luo and Zhang have shown that the identities

$$x^8y \approx x^2y, \quad xy^8 \approx xy^2, \quad x^7yx \approx xyx, \quad x^2yx \approx xyx^2, \quad xyxx \approx x^2yxx,$$
and (♦) constitute a basis for the variety $S_3$ generated by all semigroups of order three [16, Corollary 4.6]. The monoid $B_0^1$ can then be shown to belong to the variety $S_3$. It follows that results of Luo and Zhang [16] will be useful in the present proof.

It is routinely verified that the monoids $B_0^1$, $L_1^2$, and $R_1^2$ satisfy the identities $\{(\star), (\bullet)\}$. Therefore, to complete the proof, it suffices to show that any identity $w \approx w'$ satisfied by the variety $B_0^1 \lor L_1^2 \lor R_1^2$ is deducible from the identities $\{(\star), (\bullet)\}$. By Remark 3.5 and Lemma 3.6, the words $w$ and $w'$ can be chosen to be quadratic and tight. Then the words $w$ and $w'$ satisfy conditions (CF1)–(CF4) in Luo and Zhang [16, Section 4] and so are said to be in canonical form. Now the monoid $N_2^3$ is isomorphic to a submonoid of $B_0^1$ and so satisfies the identity $w \approx w'$; since the monoids $L_1^2$ and $R_1^2$ also satisfy the identity $w \approx w'$, the conditions $\text{ini}(w) = \text{ini}(w')$, $\text{fin}(w) = \text{fin}(w')$, and $\text{occ}(x, w) = \text{occ}(x, w')$ for all $x \in X$ hold by Lemma 3.2. It then follows from Luo and Zhang [16, Lemma 4.5] that the words $w$ and $w'$ are identical. Consequently, the identity $w \approx w'$ is deducible from the identities $\{(\star), (\bullet)\}$. \hfill \Box

3.3. Other required identities

**Lemma 3.7.** Let $V$ be any subvariety of $A_2^1$. 
(i) If $A_0^1 \notin V$, then $V$ satisfies the identity
\[(x^2y^2)^2 \approx x^2y^2.\]
(ii) If $A_1^1 \notin V$, then $V$ satisfies the identity
\[((x^2y)^2(yx^2)^2)^2 \approx (x^2y)^2.\]
(iii) If $R_1^1 \notin V$, then $V$ satisfies the identity
\[(x^2y)^2x^2 \approx (x^2y)^2.\]
(iv) If $L_1^1 \notin V$, then $V$ satisfies the identity
\[x^2(yx^2)^2 \approx (yx^2)^2.\]
(v) If $B_0^1 \notin V$, then $V$ satisfies one of the identities
\[x^2 \approx x,\]
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\[ x^2yx^2 \approx x^2y, \]  
\[ x^2yx^2 \approx yx^2. \]  

\begin{proof}
Part (v) follows from Almeida [1, Proposition 11.10.2]. Since parts (iii) and (iv) are dual results, it suffices to verify parts (i)–(iii). Let \( S \in \{A_0, A_2, R_2\} \). Suppose that \( S \notin \mathbb{V}_S \). Then \( S \notin \mathbb{V} \) by Lemma 2.1.

(i) If \( S = A_0 \), then \( \mathbb{V}_S \vdash (3.1) \) by Torlopoa [21].

(ii) If \( S = A_2 \), then \( \mathbb{V}_S \vdash (3.2) \) by Lee [9].

(iii) If \( S = R_2 \), then \( \mathbb{V}_S \vdash (3.3) \) by Almeida [1, Proposition 10.10.2(c)]. \( \square \)

4. Finitely based varieties generated by 2-testable monoids

\begin{proposition}
\begin{enumerate}
\item The lattice in Figure 2 coincides with \( \mathcal{L}(A_0^1 \vee L_2^1 \vee R_2^1) \).
\item The varieties in Figure 2 are precisely all finitely based varieties generated by 2-testable monoids.
\end{enumerate}
\end{proposition}

Fig. 2. The lattice of finitely based varieties generated by 2-testable monoids
In Lemma 4.2, the subvarieties of $A_0^1 \lor L_2^1 \lor R_2^1$ are partitioned into four disjoint intervals. The varieties in these intervals are then described in Lemma 4.3. Based on these results, the proof of Proposition 4.1 is given at the end of the section.

**Lemma 4.2.** The lattice $L\left( A_0^1 \lor L_2^1 \lor R_2^1 \right)$ is the disjoint union of the intervals

$I_1 = [L_2^1 \lor R_2^1, A_0^1 \lor L_2^1 \lor R_2^1],$
$I_2 = [L_2^1, A_0^1 \lor L_2^1],$
$I_3 = [R_2^1, A_0^1 \lor R_2^1],$
$I_4 = L(A_0^1).$

**Proof.** Let $V \in L\left( A_0^1 \lor L_2^1 \lor R_2^1 \right)$. Then by Lemma 3.4(i), the variety $V$ satisfies the identity $(\star)$. There are four cases.

**Case 1.** $L_2^1 \lor R_2^1 \in V$. Then $V \in I_4.$

**Case 2.** $L_2^1 \in V$ and $R_2^1 \not\in V$. Then by Lemma 3.7(iii), the variety $V$ satisfies the identity (3.3). Since 

\[ xyxy \approx xy(x^2y) \approx xy(x^2y) \approx xy^2x, \]

the variety $V$ also satisfies the identity $(\triangleright)$. Hence $V \in I_3$ by Lemma 3.4(ii).

**Case 3.** $L_2^1 \not\in V$ and $R_2^1 \in V$. By an argument that is symmetrical to Case 2, the variety $V$ satisfies the identity $(\triangleleft)$ so that $V \in I_3$ by Lemma 3.4(iii).

**Case 4.** $L_2^1, R_2^1 \not\in V$. By Cases 2 and 3, the variety $V$ satisfies the identities $(\triangleright)$ and $(\triangleleft)$. Therefore $V \in I_4$ by Lemma 3.4(iv).

**Lemma 4.3.**

(i) The varieties in the interval $I_1$ constitute the chain

$L_2^1 \lor R_2^1 \subset B_0^1 \lor L_2^1 \lor R_2^1 \subset A_0^1 \lor L_2^1 \lor R_2^1.$

(ii) The varieties in the interval $I_2$ constitute the chain

$L_2^1 \subset N_2^1 \lor L_2^1 \subset B_0^1 \lor L_2^1 \subset A_0^1 \lor L_2^1.$

(iii) The varieties in the interval $I_3$ constitute the chain

$R_2^1 \subset N_2^1 \lor R_2^1 \subset B_0^1 \lor R_2^1 \subset A_0^1 \lor R_2^1.$

(iv) The eight subvarieties of $A_0^1$ in Figure 2 constitute the interval $I_4.$
Proof. (i) Let $V \in \mathcal{I}_1$ so that $V$ satisfies the identity (\(\star\)) by Lemma 3.4(i). Suppose that $V \neq A_0^1 \vee L_1^1 \vee R_1^1$. Then $A_0^1 \notin V$ because $L_1^1, R_1^1 \in V$ by assumption. By Lemma 3.7(i), the variety $V$ satisfies the identity (3.1). Since 

\[
xyxy \approx x^2y^2x^2y^2 \quad (3.1)
\]

the variety $V$ satisfies the identity (\(\diamondsuit\)) so that $V \subseteq B_0^1 \vee L_1^1 \vee R_1^1$ by Lemma 3.4(v).

Suppose that $V \neq B_0^1 \vee L_1^1 \vee R_1^1$. Then $B_0^1 \notin V$ because $L_1^1, R_1^1 \in V$ by assumption. By Lemmas 3.2 and 3.7(v), the variety $V$ satisfies the identity (3.5). It then follows from Lemma 3.3(iii) that $V = L_2^1 \vee R_2^1$.

(ii) Let $V \in \mathcal{I}_2$ so that $V$ satisfies the identities \{ (\(\star\)), \(\bullet\) \} by Lemma 3.4(ii). Suppose that $V \neq A_0^2 \vee L_1^2$. Then $A_0^2 \notin V$ since $L_1^2 \in V$ by assumption. By the same argument in part (i), the variety $V$ satisfies the identity (\(\diamondsuit\)) so that $V \subseteq B_0^1 \vee L_1^1$ by Lemma 3.4(vi).

Suppose that $V \neq B_0^1 \vee L_1^1$. Then $B_0^1 \notin V$ because $L_1^1 \in V$ by assumption. By Lemmas 3.2(i) and 3.7(v), the variety $V$ satisfies either the identity (3.5) or the identity (3.6). But since 

\[
xy \approx x^2y^2 \quad \text{(3.5)}
\]

\[
x^2 \approx x \quad \text{(3.6)}
\]

the variety $V$ satisfies the identity $xy \approx x^2y$ in either case so that $V \subseteq N_2^1 \vee L_2^1$ by Lemma 3.3(iv).

Suppose that $V \neq N_2^1 \vee L_2^1$. Then $N_2^1 \notin V$ because $L_2^1 \in V$ by assumption. By Lemma 3.2(iii), the variety satisfies the identity $x^2 \approx x$ so that $V = L_2^1$ by Lemma 3.3(i).

(iii) This is symmetrical to part (ii).

(iv) This was established by Lee [11, Section 5].

Proof of Proposition 4.1. (i) This follows from Lemmas 4.2 and 4.3.

(ii) Let $V$ be any finitely based variety generated by 2-testable monoids. Then $V \subseteq A_0^1 \vee L_1^1 \vee R_1^1$ by Lemma 3.1. By part (i), the variety $V$ is one of the varieties in Figure 2.

5. Non-finitely based varieties generated by 2-testable monoids

Proposition 5.1. The non-finitely based varieties generated by 2-testable monoids constitute the join-semilattice in Figure 3.
Let $P$ and $Q$ be semigroups with the following multiplication tables:

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The present section requires the semigroup varieties

\[ A_0 = \mathcal{V}_S\{A_0\}, \quad B_2 = \mathcal{V}_S\{B_2\}, \quad L_2 = \mathcal{V}_S\{L_2\}, \]

\[ P = \mathcal{V}_S\{P\}, \quad Q = \mathcal{V}_S\{Q\}, \quad R_2 = \mathcal{V}_S\{R_2\}, \]

and the monoid varieties

\[ P^1 = \mathcal{V}_M\{P\}, \quad Q^1 = \mathcal{V}_M\{Q\}. \]

For any variety $\mathfrak{V}$, the dual variety of $\mathfrak{V}$ is

\[ \mathfrak{V}_\delta = \{ V \mid V \text{ is anti-isomorphic to some member of } \mathfrak{V} \}. \]

For example, $\mathcal{V}_\delta^1 = \mathcal{V}^1$ and $\mathcal{V}^1_\delta = \mathcal{V}$. The varieties $A_0^1, A_2^1, B_0^1, B_2^1, L_1^1, N_1^1$, and $\mathcal{V}$ are self-dual in the sense that they satisfy the equation $\mathcal{V}_\delta^1 = \mathcal{V}$.

Some results regarding the varieties $P^1$ and $Q^1$ are established in Subsection 5.1. The proof of Proposition 5.1 is then given in Subsection 5.2.
5.1. The varieties $P^1$ and $Q^1$

**Lemma 5.2** (Lee [12, Proposition 3.3]).

(i) $A_1^0 \lor L_2^1 = Q^1_1$.

(ii) $A_1^1 \lor R_2^1 = Q^1_1$.

Any words $w_1, \ldots, w_n$ are said to be *disjoint* if the sets $\text{con}(w_1), \ldots, \text{con}(w_n)$ are pairwise disjoint. A word of length at least two is *connected* if it cannot be written as a product of two disjoint nonempty words. Any word $w$ can be uniquely written in *natural form*, that is,

$$w = \prod_{i=1}^{n}(s_i w_i)$$

where each $s_i$ is a simple word with $s_1$ possibly being empty, each $w_i$ is a product of disjoint connected words with $w_n$ possibly being empty, and the words $s_1, w_1, \ldots, s_n, w_n$ are disjoint.

**Lemma 5.3** (Lee and Volkov [15, Proposition 3.2(ii)]). Let $w = \prod_{i=1}^{n}(s_i w_i)$ and $w' = \prod_{i=1}^{n'}(s'_i w'_i)$ be any words written in natural form. Then $B_2 \models w \approx w'$ if and only if $n = n'$, $s_i = s'_i$, and $B_2 \models w_i \approx w'_i$ for all $i$.

**Lemma 5.4** (Lee [13, Corollary 5.6(iii) and Lemma 6.2(i)]). Let $w \approx w'$ be any identity satisfied by the semigroup $B_2$. Suppose that the words $w = \prod_{i=1}^{n}(s_i w_i)$ and $w' = \prod_{i=1}^{n'}(s'_i w'_i)$ are in natural form. Then $P \models w \approx w'$ if and only if $h(w_i) = h(w'_i)$ for all $i$.

**Lemma 5.5.**

(i) $B_1^1 \lor L_2^1 = B_2^1 \lor P^1$.

(ii) $B_1^1 \lor R_2^1 = B_2^1 \lor P_3^1$.

**Proof.** By symmetry, it suffices to verify part (i). Let $w \approx w'$ be any identity satisfied by the variety $B_1^1 \lor L_2^1$. By Lemma 5.3,

$$w = \prod_{i=1}^{n}(s_i w_i) \quad \text{and} \quad w' = \prod_{i=1}^{n}(s_i w'_i)$$

when $w$ and $w'$ are written in natural form. Since the words $s_1, w_1, \ldots, s_n$, $w_n$ are disjoint, the words $s_1, w'_1, \ldots, s_n, w'_n$ are disjoint, and that $\text{ini}(w) = \text{ini}(w')$ by Lemma 3.2(i), it follows that $h(w_i) = h(w'_i)$ for all $i$. Hence by Lemma 5.4, the semigroup $P$ satisfies the identity $w \approx w'$. Since the identity $w \approx w'$ was arbitrarily chosen, $P \in V_S \left(B_1^1 \lor L_2^1\right)$. Therefore $P^1 \in B_1^1 \lor L_2^1$. 
by Lemma 2.1. Consequently, the inclusion $B_2^1 \lor P_1 \subseteq B_2^1 \lor L_2^1$ holds. The inclusion $B_2^1 \lor L_2^1 \subseteq B_2^1 \lor P_1$ holds since the monoid $L_2^1$ is isomorphic to the submonoid $\{d, e, 1\}$ of $P_1$. \hfill \square

5.2. Proof of Proposition 5.1

As observed in Subsection 3.1, the nine varieties in Figure 3 are all generated by 2-testable monoids, and it is evident from the generating monoids that the varieties form a join-semilattice. Further, these varieties contain the inherently non-finitely based monoid $B_2^1$ and so are non-finitely based. Lemma 3.7 can be used to distinguish these varieties. For instance, the identity (3.2) is satisfied by the monoids $A_0^1$, $B_2^1$, $L_2^1$, and $R_2^1$, but not by the monoid $A_2^1$. Hence $A_0^1 \neq A_0^1 \lor B_2^1 \lor L_2^1 \lor R_2^1$.

It remains to show that if $S^1$ is any non-finitely based 2-testable monoid, then it generates one of the varieties in Figure 3. Let $S = V_S\{S\}$ and $S^1 = \bigvee_M\{S^1\}$. By Theorem 1.1, the variety $S$ belongs to the interval $[B_2, A_2]$. The structure of the interval $[B_2, A_2]$, shown in Figure 4, follows from results of Lee [13, Section 5].

![Fig. 4. The interval $[B_2, A_2]$](image-url)
Remark 5.6.

(i) In Figure 4, only varieties in the interval $[B_2, A_2]$ that are required in the present section are labeled. Refer to Lee [13] for more information on the unlabeled varieties and other subvarieties of $A_2$.

(ii) The interval $[A_0 \lor B_2, B_2 \lor Q \lor Q]$ is isomorphic to the direct product of two $\omega + 2$ chains, while the interval $[B_2, B_2 \lor P \lor P]$ is isomorphic to the direct product of two $\omega + 3$ chains.

It is clear that if $S \in \{A_2, B_2, A_0 \lor B_2\}$, then $S^1 \in \{A_1^1, B_1^1, A_0^1 \lor B_2^1\}$. Therefore it suffices to consider the case when the variety $S$ belongs to one of the following subintervals of $[B_2, A_2]$:

$$J_1 = [A_0 \lor B_2 \lor L_2 \lor R_2, B_2 \lor Q \lor Q]$$
$$J_2 = [B_2 \lor L_2 \lor R_2, B_2 \lor Q \lor Q]$$
$$J_3 = [B_2 \lor L_2 \lor R_2, B_2 \lor Q \lor Q]$$
$$J_4 = [B_2 \lor L_2 \lor R_2, B_2 \lor P \lor P]$$
$$J_5 = [B_2 \lor L_2 \lor R_2, B_2 \lor P \lor P]$$
$$J_6 = [B_2 \lor L_2 \lor R_2, B_2 \lor P \lor P]$$

The following result then verifies that the variety $S^1$ coincides with one of the varieties in Figure 3.

Lemma 5.7.

(i) If $S \in J_1$, then $S^1 = A_0^1 \lor B_2^1 \lor L_2^1 \lor R_2^1$.

(ii) If $S \in J_2$, then $S^1 = A_0^1 \lor B_2^1 \lor L_2^1$.

(iii) If $S \in J_3$, then $S^1 = A_0^1 \lor B_2^1 \lor L_2^1$.

(iv) If $S \in J_4$, then $S^1 = B_2^1 \lor L_2^1 \lor R_2^1$.

(v) If $S \in J_5$, then $S^1 = B_2^1 \lor L_2^1$.

(vi) If $S \in J_6$, then $S^1 = B_2^1 \lor L_2^1$.

Proof. Part (i) holds because if $S \in J_1$, then

$$A_0^1 \lor B_2^1 \lor L_2^1 \lor R_2^1 \subseteq S^1 \subseteq B_2^1 \lor Q^1 \lor Q^1 = A_0^1 \lor B_2^1 \lor L_2^1 \lor R_2^1$$

by Lemma 5.2. Parts (ii) and (iii) hold similarly. Part (iv) holds because if $S \in J_4$, then

$$B_2^1 \lor L_2^1 \lor R_2^1 \subseteq S^1 \subseteq B_2^1 \lor P^1 \lor P^1 = B_2^1 \lor L_2^1 \lor R_2^1$$
by Lemma 5.5. Parts (v) and (vi) hold similarly. □

6. The lattice $L(B_2^1)$

Subsection 6.2 presents a chain in the lattice $L(B_2^1)$ that is isomorphic to the integers; this lattice thus violates both the ascending chain and descending chain conditions. Subsection 6.3 demonstrates that the lattice $L(B_2^1)$ contains finite anti-chains of arbitrary order and so has infinite width. The lattice $L(B_2^1)$ is also shown in Subsection 6.4 to contain non-finitely generated varieties.

For any word $w$, let $S(w)$ denote the Rees quotient monoid of $X^*$ over the ideal of all words that are not factors of $w$. Equivalently, $S(w)$ can be treated as the monoid that consists of every factor of the word $w$, together with a zero element 0, with binary operation $\cdot$ given by

$$a \cdot b = \begin{cases} ab & \text{if } ab \text{ is a factor of } w; \\ 0 & \text{otherwise.} \end{cases}$$

The empty factor, more conveniently written as 1, is the identity element of the monoid $S(w)$. Note that 0 and 1 are the only idempotents of the monoid $S(w)$.

6.1. Isoterms and Zimin words

A word $w$ is an isoterms for a semigroup $S$ if $S$ does not satisfy any nontrivial identity of the form $w \approx w'$.

**Lemma 6.1** (Jackson [5, Lemma 3.3]). Let $w$ be any word and let $M$ be any monoid. Then $w$ is an isoterms for $M$ if and only if $S(w) \in \forall_M \{ M \}$.

The Zimin words [29] are defined by $z_1 = x_1$ and $z_{n+1} = z_n x_{n+1} z_n$ for all $n \geq 1$. For each $n \geq 1$, define the monoid variety $Z_n = \forall_M \{ S(z_n) \}$.

**Lemma 6.2** (M. V. Sapir [19, Proposition 7 and Lemma 3.7]).

(i) A finite semigroup $S$ is inherently non-finitely based if and only if every Zimin word is an isoterms for $S$.

(ii) Every Zimin word is an isoterms for the monoid $B_2^1$. Consequently, the monoid $B_2^1$ is inherently non-finitely based and $Z_n \subseteq B_2^1$ for all $n \geq 1$. 
6.2. An infinite chain in $\mathcal{L}(B_1^2)$

**Lemma 6.3.** For any $n \geq 1$, the inclusions $Z_n \subset Z_{n+1} \subset B_1^2$ are proper.

**Proof.** The word $z_n$ is clearly an isoterm for the monoid $S(z_{n+1})$. Hence the inclusions $Z_n \subseteq Z_{n+1} \subseteq B_1^2$ hold by Lemmas 6.1 and 6.2(ii). It is routinely shown that the identity $z_{n+1} \approx x_1z_{n+1}$ is satisfied by the monoid $S(z_{n+1})$ but not by the monoid $B_1^2$. Therefore $Z_n \neq Z_{n+1}$. Since all Zimin words are square-free, the monoid $S(z_{n+1})$ satisfies the identity $x^2y \approx yx^2$. But the substitution $(x, y) \mapsto (ab, a)$ shows that the monoid $B_1^2$ does not satisfy the identity $x^2y \approx yx^2$. Hence $Z_{n+1} \neq B_1^2$. \qed

**Lemma 6.4.** The words $xyzxy$ and $xyzyx$ are isoterms for the monoid $S(z_3)$.

**Proof.** Suppose that some word from $\{xyzxy, xyzyx\}$ is not an isoterm for the monoid $S(z_3)$ so that this monoid satisfies some nontrivial identity $xyzxy \approx w$ or some nontrivial identity $xyzyx \approx w$. Performing the substitution $y \mapsto yx$ on the former identity and the substitution $z \mapsto zy$ on the latter identity, a nontrivial identity of the form $xyzzyzx \approx w'$ is obtained in either case. Hence the word $xyzzyzx$ is not an isoterm for the monoid $S(z_3)$. But this is impossible since the words $xyzzyzx$ and $z_3$ are the same up to a permutation on the letters in $X$. \qed

**Proposition 6.5.** The lattice $\mathcal{L}(B_1^2)$ contains a chain that is isomorphic to the integers.

**Proof.** In the presence of Lemma 6.3, it suffices to show that the lattice $\mathcal{L}(Z_3)$ contains an infinite decreasing chain. It follows from Lemmas 6.1 and 6.4 that the varieties $W = V_M \{ S(xyzxy) \}$ and $W' = V_M \{ S(xyzyx) \}$ are contained in $Z_3$. Jackson and O. Sapir [7, Section 5] proved that the varieties $W$ and $W \lor W'$ are non-finitely based and finitely based respectively. Consequently, the subinterval $[W, W \lor W']$ of $\mathcal{L}(Z_3)$ contains the required infinite decreasing chain. \qed

6.3. Finite anti-chains in $\mathcal{L}(B_1^2)$ of arbitrary order

**Lemma 6.6.** Let $w_0, \ldots, w_n \in \{ y_1y_2, y_2y_1 \}$ with $n \geq 1$. Then the word

$$ w = w_0 \prod_{i=1}^{n} (h_iw_i) $$

is an isoterm for the monoid $S(z_{m+2})$ for any $m$ such that $2^m > n$. 

Proof. Without loss of generality, assume that \( w_0 = y_1 y_2 \). By definition, the word \( z_2 = x_1 x_2 x_1 \) is a factor of the word \( z_{m+2} \). More specifically, it is routinely shown by induction on \( m \) that

\[
z_{m+2} = z_2 \prod_{i=1}^{2^m-1} (t_i z_2)
\]

for some \( t_1, \ldots, t_{2^m-1} \in \{ x_3, \ldots, x_{m+2} \} \). Let \( \varphi \) denote the substitution given by \( y_1 \varphi = x_1, y_2 \varphi = x_2 \), and for each \( i \in \{ 1, \ldots, n \} \),

\[
h_i \varphi = \begin{cases} t_i, & \text{if } w_{i-1} = y_2 y_1 \text{ and } w_i = y_1 y_2; \\ x_1 t_i, & \text{if } w_{i-1} = y_1 y_2 \text{ and } w_i = y_1 y_2; \\ t_i x_1, & \text{if } w_{i-1} = y_2 y_1 \text{ and } w_i = y_2 y_1; \\ x_1 t_i x_1, & \text{if } w_{i-1} = y_1 y_2 \text{ and } w_i = y_1 y_2. \end{cases}
\]

Then \( w \varphi \) is a prefix of the word \( z_{m+2} \). (For example, consider the word

\[
w = y_1 y_2 h_1 y_1 y_2 h_2 y_2 y_1 h_3 y_2 y_1 h_4 y_1 y_2 h_5 y_2 y_1
\]

with \( n = 5 \). Since \( 2^5 > 5 \), it suffice to choose \( m = 3 \). Then

\[
w \varphi = x_1 x_2 \cdot h_1 \varphi \cdot x_1 x_2 \cdot h_2 \varphi \cdot x_2 x_1 \cdot h_3 \varphi \cdot x_2 x_1 \cdot h_4 \varphi \cdot x_1 x_2 \cdot h_5 \varphi \cdot x_2 x_1
\]

is a prefix of the word \( z_5 \). Let \( s \in X^* \) be such that \( z_{m+2} = (w \varphi) s \).

Working toward a contradiction, suppose that the word \( w \) is not an isoterm for the monoid \( S(z_{m+2}) \) so that this monoid satisfies a nontrivial identity of the form \( w \approx w' \). Since every simple word is an isoterm for the monoid \( S(z_{m+2}) \), it follows that \( w' = w_0' \prod_{i=1}^{2^m-1} (h_i w_i') \) for some \( w_0', \ldots, w_n' \in \{ y_1, y_2 \}^* \). It is then easily shown that \( w \varphi \neq w' \varphi \). Now the monoid \( S(z_{m+2}) \) satisfies the nontrivial identity \( (w \varphi) s \approx (w' \varphi) s \) where \( (w \varphi) s = z_{m+2} \), and this is impossible.

Proposition 6.7. For each \( m \geq 0 \), the lattice \( \mathcal{L}(Z_{m+2}) \) has width at least \( 2^m \). Consequently, the lattice \( \mathcal{L}(B_2^1) \) contains an anti-chain of each finite order.

Proof. The result clearly holds if \( m = 0 \). Since the subvarieties \( \mathbb{W} \) and \( \mathbb{W}' \) of \( Z_3 \) in the proof of Proposition 6.5 are incomparable, the lattice \( \mathcal{L}(Z_3) \) has width at least two. Therefore it suffices to assume that \( m \geq 2 \). Let \( 2^m = n + 1 \) (so that \( n \geq 3 \)) and let \( y = y_1 y_2 \prod_{i=1}^{n} (h_i y_1 y_2) \). For each
$j \in \{1, \ldots, n + 1\}$, replace the $j$th factor $y_1 y_2$ in the word $y$ by $y_2 y_1$, and denote the resulting word by $y_j$, that is,

$$y_1 = y_2 y_1 \cdot h_1 y_1 y_2 \cdot h_2 y_1 y_2 \cdots h_{n-1} y_1 y_2 \cdot h_n y_1 y_2,$$

$$y_2 = y_1 y_2 \cdot h_1 y_2 y_1 \cdot h_2 y_2 y_1 \cdots h_{n-1} y_2 y_1 \cdot h_n y_2 y_1,$$

$$\vdots$$

$$y_{n+1} = y_1 y_2 \cdot h_1 y_1 y_2 \cdot h_2 y_1 y_2 \cdots h_{n-1} y_1 y_2 \cdot h_n y_2 y_1.$$

By Lemmas 6.1 and 6.6, the varieties $V_M \{S(y_1)\}, \ldots, V_M \{S(y_{n+1})\}$ are contained in the variety $Z_{m+2}$; these $n + 1$ subvarieties of $Z_{m+2}$ are pairwise incomparable since it is routinely checked that $S(y_j) \models y_k \approx y$ if and only if $j \neq k$.

6.4. Non-finitely generated varieties in $\mathcal{L}(B_2^1)$

**Lemma 6.8** (M. V. Sapir [18, Theorem 2]). A finite aperiodic monoid $M$ is inherently non-finitely based if and only if $B_2^1 \in V_M \{M\}$.

Recall from Lemma 6.3 that $Z_1 \subset Z_2 \subset \cdots \subset B_2^1$. Since the identity $x^2 y \approx y x^2$ is satisfied by any monoid $S(z_n)$ but not by the monoid $B_2^1$, the complete join

$$Z_\infty = Z_1 \lor Z_2 \lor \cdots$$

is a proper subvariety of the variety $B_2^1$.

**Proposition 6.9.** Let $\mathcal{V}$ be any aperiodic monoid variety such that $Z_\infty \subseteq \mathcal{V}$ and $B_2^1 \notin \mathcal{V}$. Then the variety $\mathcal{V}$ is non-finitely generated. Consequently, every proper subvariety of $B_2^1$ that contains $Z_\infty$ is non-finitely generated.

**Proof.** Suppose that $\mathcal{V} = V_M \{M\}$ for some finite monoid $M$. Since $Z_\infty \subseteq \mathcal{V}$, it follows from Lemma 6.1 that every Zimin word is an isoterm for the monoid $M$. By Lemma 6.2(i), the monoid $M$ is inherently non-finitely based. The contradiction $B_2^1 \in \mathcal{V}$ then follows from Lemma 6.8. \qed
7. Open questions

7.1. Varieties in Figure 1

Let $\mathcal{X}$ denote the join-semilattice in Figure 1. Let $\mathcal{X} = \mathcal{F} \cup \mathcal{N}$ where $\mathcal{F}$ consists of finitely based varieties in $\mathcal{X}$, and $\mathcal{N}$ consists of non-finitely based varieties in $\mathcal{X}$. Recall from Propositions 4.1 and 5.1 that $\mathcal{F}$ coincides with the lattice $\mathcal{L}(A_1^0 \lor L_1^2 \lor R_1^2)$ and that $\mathcal{N}$ is a join-semilattice.

**Question 7.1.** Is the join-semilattice $\mathcal{N}$ a lattice?

Let $V, V' \in \mathcal{X}$. By Proposition 4.1 and the 2-testable monoids given in Subsection 3.1 that generate the varieties in $\mathcal{X}$, it is easily verified that $V \lor V' \in \mathcal{X}$. Since $\mathcal{F} = \mathcal{L}(A_1^0 \lor L_1^2 \lor R_1^2)$, either $V \in \mathcal{F}$ or $V' \in \mathcal{F}$ implies that $V \cap V' \in \mathcal{F} \subset \mathcal{X}$. It follows that an affirmative answer to Question 7.1 implies that the join-semilattice $\mathcal{X}$ is a lattice.

7.2. The interval $[B_1^2, A_2^2]$  

As commented in Section 1, the task of identifying all varieties in the interval $[B_1^2, A_2^2]$ is hindered by the presence of non-finitely based varieties within it. But the complete description of the interval $[B_2^2, A_2^2]$ (see Figure 4) inspires the conjecture of bases of some varieties within the variety $A_2^2$.

**Question 7.2.** Which of the following equations hold?

\begin{align}
(7.1) & \quad B_1^2 = A_2^2 \cap \{x^2 y^2 \approx y^2 x^2\}, \\
(7.2) & \quad B_1^2 \lor L_2^2 = A_2^2 \cap \{x^2 y^2 x^2 \approx x^2 y^2\}, \\
(7.3) & \quad B_1^2 \lor R_2^2 = A_2^2 \cap \{x^2 y^2 x^2 \approx y^2 x^2\}, \\
(7.4) & \quad B_1^2 \lor L_2^2 \lor R_2^2 = A_2^2 \cap \{x^2 y^2 x^2 y^2 \approx x^2 y^2\}, \\
(7.5) & \quad A_0^1 \lor B_1^2 = A_2^2 \cap \{x^2 y^2 x^2 \approx y^2 x^2 y^2\}, \\
(7.6) & \quad A_0^1 \lor B_1^2 \lor L_2^2 = A_2^2 \cap \{x^2 y^2 x^2 y^2 \approx x^2 y^2 x^2\}, \\
(7.7) & \quad A_0^1 \lor B_1^2 \lor R_2^2 = A_2^2 \cap \{x^2 y^2 x^2 y^2 \approx y^2 x^2 y^2\}, \\
(7.8) & \quad A_0^1 \lor B_1^2 \lor L_2^2 \lor R_2^2 = A_2^2 \cap \{x^2 y^2 x^2 z^2 x^2 \approx x^2 y^2 z^2 x^2\}.
\end{align}

It is routinely shown that if (7.1)–(7.7) hold, then the answer to Question 7.1 is affirmative.
Question 7.3. Is every variety in the interval \([B_2, A_2]\) of the form 
\(A_2 \cap \Sigma\) for some finite set \(\Sigma\) of identities?

7.3. Number of subvarieties

Trahtman [24] proved that the semigroup variety \(A_2 = V_S\{A_2\}\) contains continuum many subvarieties while Jackson [4] later proved that the smaller variety \(B_2 = V_S\{B_2\}\) also has the same property. However, the only monoids that belong to these subvarieties are semilattices. Therefore no conclusion on the number of subvarieties of the monoid varieties \(A_2\) and \(B_2\) can be drawn from the aforementioned results of Jackson and Trahtman.

Question 7.4. Does any of the monoid varieties \(A_2\) and \(B_2\) contain continuum many subvarieties?

Jackson and McKenzie [6] presented a monoid of order 56 that generates a monoid variety with continuum many subvarieties. An affirmative answer to Question 7.4 thus provides a significantly smaller example.

7.4. Non-finitely generated varieties

It follows from Proposition 6.9 that every variety in the interval \([\mathbb{Z}_\infty, A_2 \cap [x^2y \approx yx^2]]\) is non-finitely generated.

Question 7.5. Which of the inclusions \(\mathbb{Z}_\infty \subseteq B_2 \cap [x^2y \approx yx^2] \subseteq A_2 \cap [x^2y \approx yx^2]\) is proper?

Note that if (7.1) holds, then \(B_2 \cap [x^2y \approx yx^2] = A_2 \cap [x^2y \approx yx^2]\).

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REFERENCES


