FINITE BASIS PROBLEM FOR THE DIRECT PRODUCT OF SOME J-TRIVIAL MONOID WITH GROUPS OF FINITE EXPONENT

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ABSTRACT. It is proved that the direct product of the J-trivial monoid $S(xyx)$ with any noncommutative group of finite exponent is non-finitely based. This result provides new, simpler examples of two finitely based finite monoids for which their direct product is non-finitely based. It follows that the direct product of the monoid $S(xyz)$ with any group of finite exponent is finitely based if and only if the group is commutative.

1. Introduction

For any set $W$ of words over a countably infinite alphabet $X$, let $S(W)$ denote the Rees quotient of the free monoid $X^*$ modulo the ideal of all words that are not factors of any word in $W$. Equivalently, $S(W)$ can be treated as the monoid that consists of every factor of every word in $W$, together with a zero element $0$, with binary operation $\cdot$ given by

$$U \cdot V = \begin{cases} UV, & \text{if } UV \text{ is a factor of some word in } W; \\ 0, & \text{otherwise.} \end{cases}$$

The empty factor, more conveniently written as $1$, is the unit element of the monoid $S(W)$. If $W = \{W_1, \ldots, W_n\}$, then write $S(W) = S(W_1, \ldots, W_n)$. Each $S(W)$ is a J-trivial monoid with $0$ and $1$ as its only idempotent elements.

The class $R$ of Rees quotient monoids of the form $S(W)$ constitutes a significant source of examples in the study of the finite basis problem for semigroups and monoids. For instance, one of the first published examples of non-finitely based finite semigroups, due to Perkins [10] in the 1960s, is the monoid $S(xyx, xyzyx, xzyxy, x^2z)$ of order 25. Recent investigations [3–8, 12] shed more light on the finite basis problem for monoids in $R$. In particular, results from Jackson [3] and Jackson and Sapir [5] demonstrate that with respect to the finite basis property, the class $R$ behaves as irregularly as the class of all finite semigroups. Refer to the survey by Volkov [14] for more information on the finite basis problem for monoids in $R$ and for finite semigroups in general.

The present article is concerned with the finite basis problem for the class of monoids. One of the aforementioned irregularities of the class $R$ demonstrated by Jackson and Sapir is the non-closure of its finitely based finite monoids under the formation of direct products.

**Theorem 1** (Jackson [4] and Jackson and Sapir [5]). There exist finitely based finite monoids $M_1$ and $M_2$ for which the direct product $M_1 \times M_2$ is non-finitely based. Examples of these monoids include

- $M_1 = S(xyhxty)$ and $M_2 = S(xhytxy)$;
- $M_1 = S(x^2y^2x, xyxyx, xy^2x^2)$ and $M_2 = S(x^2y^2, xyxy, xy^2x, yx^3y)$;
- $M_1 = S(xyzyx, xyzxy)$ and $M_2 = S(x^ny^n)$ for any $n \geq 2$.

**Remark 2** Although there exist examples of two finitely based finite semigroups with non-finitely based direct product [11, 13], none of these examples implies examples of monoids $M_1$ and $M_2$ that satisfy Theorem 1.
One objective of the present article is to expand the list in Theorem 1 with new, simpler examples of monoids $M_1$ and $M_2$. This is achieved by a general result concerning the Rees quotient monoid $S(xyx)$ of order seven.

**Theorem 3** The direct product of the monoid $S(xyx)$ with any noncommutative group of finite exponent is non-finitely based.

The monoid $S(xyx)$ was first investigated in detail by Jackson [1, 2], who proved that it is a finitely based monoid that generates a semigroup variety with continuum many subvarieties. He later showed that the monoid variety generated by $S(xyx)$ contains only five subvarieties [4].

Now the well-known theorem of Oates and Powell states that all finite groups are finitely based [9]. Therefore Theorem 3 provides new, easily described monoids $M_1$ and $M_2$ that satisfy Theorem 1; the simplest case occurs when $M_1$ is the monoid $S(xyx)$ and $M_2$ is the symmetric group of order six. Note that most of the monoids in Theorem 1 are substantially larger:

- $|S(xyhxty)| = |S(xhytxy)| = 21$;
- $|S(x^2y^2x, xyxy, xy^2x)| = 22$ and $|S(x^2y^2, xyxy, xy^2x, yx^3y)| = 21$;
- $|S(xyzyx, xyzyx)| = 21$ and $|S(x^n y^n)| = (n + 1)^2 + 1$.

In contrast with Theorem 3, the direct product of the monoid $S(xyx)$ with any commutative group is finitely based [7]. Therefore the finite basis problem for the direct product of $S(xyx)$ with any group of finite exponent has a complete solution.

**Corollary 4** For any group $G$ of finite exponent, the direct product $S(xyx) \times G$ is finitely based if and only if $G$ is commutative.

Notation and background material are given in Section 2. Some restrictions on identities satisfied by the monoid $S(xyx)$ and by noncommutative groups are established in Section 3. The proof of Theorem 3 is then given in Section 4.

### 2. Preliminaries

Let $\mathcal{X}$ be a countably infinite alphabet throughout. For any subset $\mathcal{Y}$ of $\mathcal{X}$, let $\mathcal{Y}^+$ and $\mathcal{Y}^*$ denote the free semigroup and free monoid over $\mathcal{Y}$, respectively. Elements of $\mathcal{X}$ are called letters and elements of $\mathcal{X}^*$ are called words. For any word $W$,

- the number of occurrences of a letter $x$ in $W$ is denoted by $\text{occ}(x, W)$;
- a letter $x$ is simple in $W$ if $\text{occ}(x, W) = 1$;
- the set of simple letters of $W$ is denoted by $\text{sim}(W)$;
- the content of $W$, denoted by $\text{con}(W)$, is the set of letters occurring in $W$.

An identity is written as $U \approx V$ where $U, V \in \mathcal{X}^+$. An identity $U \approx V$ is balanced if $\text{occ}(x, U) = \text{occ}(x, V)$ for all $x \in \mathcal{X}$. A monoid $M$ satisfies an identity $U \approx V$ if, for any substitution $\varphi : \mathcal{X} \rightarrow M$, the elements $U\varphi$ and $V\varphi$ of $M$ coincide. A set $\Sigma$ of identities satisfied by a monoid $M$ is a basis of $M$ if $\Sigma$ implies every identity satisfied by $M$. A monoid is finitely based if it possesses a finite basis.

For any word $W$ and any set $\mathcal{Y}$ of letters, let $W[\mathcal{Y}]$ denote the word obtained from $W$ by retaining the letters from $\mathcal{Y}$. It is easily shown that if a monoid satisfies an identity $U \approx V$, then it also satisfies the identity $U[\mathcal{Y}] \approx V[\mathcal{Y}]$ for any $\mathcal{Y} \subseteq \mathcal{X}$.
3. Restrictions on identities

For each $n \geq 1$, define the word

$$J_n = \left( \prod_{i=1}^{n} (x_i h_i) \right) y \left( \prod_{i=1}^{n} (x_i z_i) \right) y \left( \prod_{i=1}^{n} (t_i z_i) \right)$$

$$= x_1 h_1 \cdots x_n h_n y x_1 z_1 \cdots x_n z_n y t_1 \cdots t_n z_n.$$

The words $J_n$ were first employed by Jackson [4, proof of Lemma 5.5] to prove that the monoid $S(x y x y)$ is non-finitely based.

**Lemma 5** (Jackson [4, Lemmas 3.3 and 4.4]). Let $U \approx V$ be any identity satisfied by the monoid $S(x y x y)$. Then

(i) $\text{sim}(U) = \text{sim}(V)$ and $\text{con}(U) = \text{con}(V)$;

(ii) $U[x, y] = xy$ if and only if $V[x, y] = xy$;

(iii) $U[x, y] = xyx$ if and only if $V[x, y] = xyx$.

**Lemma 6** Let $J_n \approx W$ be any identity satisfied by the monoid $S(x y x y)$. Then

(i) $W[x_i, h_j] = x_i h_j x_i$ if and only if $i \leq j$;

(ii) $W[t_i, z_j] = z_j t_i z_j$ if and only if $i \leq j$;

(iii) $W[x_i, t_j] \neq x_i t_i x_i$ and $W[h_i, z_j] \neq z_j h_j z_j$ for any $i$ and $j$;

(iv) $W[h_i, y] \neq h_i y h_i$ and $W[y, t_i] \neq y t_i y$ for any $i$.

**Proof.** This follows from Lemma 5 (iii) because

(i) $J_n[x_i, h_j] = x_i h_j x_i$ when $i \leq j$ and $J_n[x_i, h_j] = h_j x_i^2$ when $i > j$;

(ii) $J_n[t_i, z_j] = z_j t_i z_j$ when $i \leq j$ and $J_n[t_i, z_j] = z_j^2 t_i$ when $i > j$;

(iii) $J_n[x_i, t_j] = x_i^2 t_j$ and $J_n[h_i, z_j] = h_i z_j^2$ for any $i$ and $j$;

(iv) $J_n[h_i, y] = h_i y^2$ and $J_n[y, t_i] = y^2 t_i$ for any $i$.

**Lemma 7** Let $U \approx V$ be any identity satisfied by any noncommutative group with $U[x, y] = x y x y$. Then $V[x, y] \notin \{x^2 y^2, x y x^2, y x^2 y\}$.

**Proof.** Any group that satisfies any of the following identities is commutative: $x y x y \approx x^2 y^2$, $x y x x \approx x y x^2 x$, and $x y x y \approx y x^2 y$. □

**Lemma 8** Let $J_n \approx W$ be any identity satisfied by a noncommutative group. Then

(i) $W[x_i, y] \neq x_i^2 y^2$ and $W[y, z_i] \neq y^2 z_i^2$ for any $i$;

(ii) $W[x_i, z_j] \neq x_i z_j x_i z_j$ when $i \leq j$;

(iii) $W[x_i, z_j] \neq x_i^2 z_j^2$ when $i > j$.

**Proof.** This follows from Lemma 7 because

(i) $J_n[x_i, y] = x_i y x_i y$ and $J_n[y, z_i] = y z_i y z_i$ for any $i$;

(ii) $J_n[x_i, z_j] = x_i^2 z_j^2$ when $i \leq j$;

(iii) $J_n[x_i, z_j] = x_i z_j x_i z_j$ when $i > j$.

**Lemma 9** Suppose that $J_n \approx W$ is any balanced identity satisfied by the monoid $S(x y x y) \times G$, where $G$ is any noncommutative group. Then $J_n \approx W$.

**Proof.** Since $J_n[h_i, t_i | 1 \leq i \leq n] = h_1 \cdots h_n t_1 \cdots t_n$, it follows from Lemma 5 (ii) that $W[h_i, t_i | 1 \leq i \leq n] = h_1 \cdots h_n t_1 \cdots t_n$. Therefore by Lemma 6 (i)–(iii),

(a) $W[x_i, h_i, t_i, z_i | 1 \leq i \leq n] = x_i h_i \cdots x_n h_n U t_1 z_1 \cdots t_n z_n$ for some $U \in \{x_1, z_1, \ldots, x_n, z_n\}$ such that $\text{occ}(x_i, U) = \text{occ}(z_i, U) = 1$ for all $i$.

If $U[x_i, x_{i+1}] = x_i \cdots x_{i+1}$ for some $i$, then $W[x_i, x_{i+1}] = x_i x_{i+1}^2 x_i$; whence Lemma 7 is violated because $J_n[x_i, x_{i+1}] = x_i x_{i+1}^2 x_i$.

Therefore $J_n[x_i, x_{i+1}] = x_i x_{i+1}$ for all $i$ so that

(b) $U[x_1, \ldots, x_n] = x_1 \cdots x_n$.

Similarly,

(c) $U[z_1, \ldots, z_n] = z_1 \cdots z_n$.  

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Hence by (a)–(c) and Lemma 8 (ii)–(iii),

\[ W[x_i, h_i, t_i, z_i | 1 \leq i \leq n] = x_1h_1 \cdots x_nh_nx_1z_1 \cdots x_nz_nh_1z_1 \cdots t_nz_n. \]

Since the identity \( J_n \approx W \) is balanced, \( \text{occ}(y, W) = 2 \). It now follows from Lemmas 6 (iv) and 8 (i) that \( W = J_n \). □

4. Proof of Theorem 3

Suppose that \( G \) is any noncommutative group of finite exponent \( m \). Let \( J'_n \) denote the word obtained from \( J_n \) by replacing the first \( y \) with \( y^{m+1} \):

\[ J'_n = \left( \prod_{i=1}^{n} (x_i h_i) \right) y^{m+1} \left( \prod_{i=1}^{n} (x_i z_i) \right) y \left( \prod_{i=1}^{n} (t_i z_i) \right). \]

Lemma 10 For each \( n \geq 1 \), the identity \( J_n \approx J'_n \) is satisfied by the monoid \( S(xy) \times G \).

Proof. Let \( \varphi : \mathcal{X} \to S(xy) \) be any substitution. Since the first and last occurrences of \( y \) in \( J_n \) and in \( J'_n \) do not sandwich any simple letters, it is easily seen that if \( y\varphi \neq 1 \), then \( J_n\varphi \) and \( J'_n\varphi \) are not factors of \( xy \) so that \( J_n\varphi = 0 = J'_n\varphi \) in \( S(xy) \). If \( y\varphi = 1 \), then clearly \( J_n\varphi = J'_n\varphi \) in \( S(xy) \). Therefore the monoid \( S(xy) \) satisfies the identity \( J_n \approx J'_n \).

Since the group \( G \) has exponent \( m \), it satisfies the identity \( y^{m+1} \approx y \) and so also the identity \( J_n \approx J'_n \). □

Seeking a contradiction, suppose that the monoid \( S(xy) \times G \) is finitely based, say with finite basis \( \Sigma \) of identities. Then there exists \( n \geq 1 \) such that each identity \( U \approx V \) in \( \Sigma \) involves at most \( n \) distinct letters, that is, \( |\text{con}(UV)| \leq n \). It is shown in Lemma 11 below that no such identity can be used to convert the word \( J_n \) into a different word. Therefore the identity \( J_n \approx J'_n \) is not implied by \( \Sigma \) and so is not satisfied by the monoid \( S(xy) \times G \), contradicting Lemma 10.

Lemma 11 Let \( U \approx V \) be any identity satisfied by the monoid \( S(xy) \times G \) with \( |\text{con}(UV)| \leq n \). Suppose that there exist words \( P, Q \in \mathcal{X}^* \) and an endomorphism \( \varphi : \mathcal{X}^* \to \mathcal{X}^* \) such that \( J_n = P(U\varphi)Q \). Then \( J_n = P(V\varphi)Q \).

Proof. The lemma clearly holds if \( U\varphi = 1 \), so assume that \( U\varphi \neq 1 \). By Lemma 5 (i),

(a) \( \text{sim}(U) = \text{sim}(V) \) and \( \text{con}(U) = \text{con}(V) \).

Further, the word \( U\varphi \) is a nonempty factor of \( J_n \) and \( \text{occ}(x, J_n) \leq 2 \) for all \( x \in \mathcal{X} \). Therefore

(b) if \( x \in \text{con}(U) \) and \( x\varphi \neq 1 \), then \( \text{occ}(x, U) \leq 2 \).

For the remainder of this proof, it is shown that

(†) if \( x \in \text{con}(U) \) and \( x\varphi \neq 1 \), then \( \text{occ}(x, U) = \text{occ}(x, V) \).

It follows that \( P(U\varphi)Q \approx P(V\varphi)Q \) is a balanced identity satisfied by the monoid \( S(xy) \times G \). Consequently, \( J_n = P(U\varphi)Q = P(V\varphi)Q \) by Lemma 9.

Let \( x \in \text{con}(U) \) be such that \( x\varphi \neq 1 \). If the letter \( x \) is simple in \( U \), then \( \text{occ}(x, U) = 1 \) by (a) so that (†) holds. Therefore it suffices to further assume that the letter \( x \) is non-simple in \( U \), whence \( \text{occ}(x, U) = 2 \) by (b). Thus \( U = U_1xU_2xU_3 \) for some \( U_1, U_2, U_3 \in \mathcal{X}^* \) with \( x \notin \text{con}(U_1U_2U_3) \), and

\[ J_n = P(U_1\varphi)(x\varphi)(U_2\varphi)(x\varphi)(U_3\varphi)Q. \]
Now it is easily seen that

(c) if \( J_n = \cdots W \cdots W \cdots \) for some \( W \in \mathcal{A}^+ \), then \( W \in \{x_1, \ldots, x_n, y, z_1, \ldots, z_n\} \). Therefore the factor \( x\varphi \) of \( J_n \) coincides with one of \( x_1, \ldots, x_n, y, z_1, \ldots, z_n \). It follows that

(d) the factor \( U_2\varphi \) of \( J_n \) contains \( 2n \) distinct letters.

Since \( |\text{con}(U_1xU_2xU_3)| = |\text{con}(U)| \leq n \), the word \( U_2 \) contains fewer than \( n \) distinct letters. If each letter in \( U_2 \) is mapped by \( \varphi \) to either 1 or a single letter, then \( |\text{con}(U_2\varphi)| \leq |\text{con}(U)| \leq n \) violates (d). Therefore there exists some letter in \( U_2 \), say \( z \), such that \( z\varphi \) is neither 1 nor a single letter, that is,

(e) the factor \( z\varphi \) of \( J_n \) contains at least two letters.

Now write \( U_2 = U'_2zU''_2 \) for some \( U'_2, U''_2 \in \mathcal{A}^* \) so that \( U = U_1xU'_2zU''_2xU_3 \). If the letter \( z \) is non-simple in \( U \), then

\[ J_n = P(U\varphi)Q = P \cdots z\varphi \cdots z\varphi \cdots Q, \]

which is impossible in view of (c) and (e). The letter \( z \) is thus simple in \( U \), whence \( U[x, z] = xzx \). Lemma 5 (iii) then implies that \( V[x, z] = xzx \). Therefore \( \text{occ}(x, U) = 2 = \text{occ}(x, V) \) and (†) holds. \( \square \)

References


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