NON-SPECHT VARIETY GENERATED BY AN INVOLUTION SEMIGROUP OF ORDER FIVE

EDMOND W. H. LEE

Dedicated to the 65th birthday of Professor Mikhail V. Volkov

Abstract. The non-orthodox 0-simple semigroup $A_2$ of order five admits a unary operation under which it is an involution semigroup. It is known that $A_2$ generates a Specht variety of semigroups. In contrast, it is shown that as an involution semigroup, $A_2$ generates a non-Specht variety.

1. Introduction

1.1. The semigroup $A_2$. The non-orthodox 0-simple semigroup

$$A_2 = \langle a, e \mid a^2 = 0, aea = a, e^2 = eae = e \rangle$$

of order five plays a distinguished role in the theory of semigroups and especially in the study of Rees–Sushkevich varieties [12,13,26,30,32,33], that is, subvarieties of periodic varieties generated by completely 0-simple semigroups. The variety $V_{A_2}$ generated by $A_2$ contains the well-known Brandt semigroup

$$B_2 = \langle a, b \mid a^2 = b^2 = 0, aba = a, bab = b \rangle$$

of order five. The semigroups $A_2$ and $B_2$ are essential to the construction of many examples with extreme varietal and equational properties [6,7,28,29,34,38,39,42]. Refer to the surveys by Shevrin et al. [35], Shevrin and Volkov [36], and Volkov [43] for more information on these semigroups.

<table>
<thead>
<tr>
<th>$A_2$</th>
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Table 1. Multiplication tables of the semigroups $A_2$ and $B_2$

Up until the 1990s, only a few results on the varieties $V_{A_2}$ and $V_{B_2}$ were established, for example, these varieties are finitely based [37, 40] and finitely universal [41] in the sense that their lattices of subvarieties embed a copy of every finite lattice. After a decade of intense investigation in the 2000s [14–16,18–20,27,31,44], the subvarieties of $V_{A_2}$ became better understood. Most crucially, the finite basis problem for these varieties has been completely solved [19].

Theorem 1. The variety $V_{A_2}$ is a Specht variety, that is, a variety whose subvarieties are all finitely based. Consequently, the lattice of subvarieties of $V_{A_2}$ is countably infinite and satisfies the descending chain condition.

2000 Mathematics Subject Classification. 20M07.

Key words and phrases. Semigroup, involution, variety, finitely based, Specht variety.
In particular, the interval $[\mathbf{V}B_2, \mathbf{V}A_2]$ is a distributive lattice [20].

![Diagram](image)

**Figure 1.** The interval $[\mathbf{V}B_2, \mathbf{V}A_2]$

1.2. **Main result.** A unary semigroup $\langle S, \ast \rangle$ that satisfies the equations

$$(x^\ast)^\ast \approx x \quad \text{and} \quad (xy)^\ast \approx y^\ast x^\ast$$

is called an *involution semigroup* or $\ast$-*semigroup*. The main algebra of the present article is the $\ast$-*semigroup* $\langle A_2, \ast \rangle$ with the transposition $ae \leftrightarrow ca$ as its unary operation, that is, $\ast$ interchanges $ae$ and $ca$ and fixes all other elements:

$$0^\ast = 0, \quad a^\ast = a, \quad e^\ast = e, \quad (ae)^\ast = ca, \quad \text{and} \quad (ca)^\ast = ac.$$

The $\ast$-*semigroup* $\langle A_2, \ast \rangle$ is the only one with $A_2$ as its semigroup reduct. In contrast, the semigroup $B_2$ is the reduct of the $\ast$-*semigroups* $\langle B_2, \ast \rangle$ and $\langle B_2, \circ \rangle$ with transpositions $ab \leftrightarrow ba$ and $a \circ \rightarrow b$, respectively. The varieties $\mathbf{V}(B_2, \circ)$ and $\mathbf{V}(A_2, \ast)$ are incomparable in the lattice of varieties of $\ast$-*semigroups* while $\mathbf{V}(B_2, \ast)$ is a subvariety of $\mathbf{V}(A_2, \ast)$ [3, proof of Corollary 2.8].

Similar to the semigroups $A_2$ and $B_2$, the $\ast$-*semigroups* $\langle A_2, \ast \rangle$, $\langle B_2, \ast \rangle$, and $\langle B_2, \circ \rangle$ are also involved in the construction of a number of examples with extreme varietal and equational properties [1–3, 10, 11, 23, 24]. The variety $\mathbf{V}(B_2, \circ)$ is a Specht variety that contains only three subvarieties [9], but each of the varieties $\mathbf{V}(A_2, \ast)$ and $\mathbf{V}(B_2, \ast)$ contains at least infinitely many subvarieties [24]. Besides these results, not much is known about the subvarieties of $\mathbf{V}(A_2, \ast)$ or $\mathbf{V}(B_2, \ast)$.

In view of the small number of elements in the semigroup $A_2$, it seems plausible that the difference between the subvarieties of $\mathbf{V}(A_2, \ast)$ and of $\mathbf{V}A_2$, if any, will not be too significant. The present article refutes this optimism by demonstrating that the lattice of subvarieties of $\mathbf{V}(A_2, \ast)$ does not bear much resemblance to that of $\mathbf{V}A_2$ (cf. Theorem 1 and Figure 1).
**Theorem 2.** The interval \([V(B_2, *), V(A_2, *)]\) violates the descending chain condition. Consequently, the variety \(V(A_2, *)\) is not a Specht variety.

Some background information and results are first established in Section 2. The desired infinite descending chain in the interval \([V(B_2, *), V(A_2, *)]\) is then exhibited in Section 3.

1.3. **Open problems.** There is a large disparity in the number of results presently available for subvarieties of \(V.A_2\) and those of \(V(A_2, *)\); see, for example, Table 2. Most notably, the finite basis property of the varieties \(V(A_2, *)\) and \(V(B_2, *)\) remains elusive even though the varieties \(V.A_2\) and \(V.B_2\) have long been shown to be finitely based. In general, there exist finite \(*\)-semigroups \(\langle S, * \rangle\) where the varieties \(V.S\) and \(V(S, *)\) are not simultaneously finitely based [8, 22, 25].

<table>
<thead>
<tr>
<th></th>
<th>(VA_2)</th>
<th>(VB_2)</th>
<th>(V(A_2, *))</th>
<th>(V(B_2, *))</th>
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</thead>
<tbody>
<tr>
<td>Finitely based</td>
<td>Yes</td>
<td>Yes</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>Specht</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
<td>?</td>
</tr>
<tr>
<td>(2^{\aleph_0}) subvarieties</td>
<td>No</td>
<td>No</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>Finitely universal</td>
<td>Yes</td>
<td>Yes</td>
<td>?</td>
<td>?</td>
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</table>

**Table 2.** Some unknown properties of \(V(A_2, *)\) and \(V(B_2, *)\)

Determining any of the unknown properties in Table 2 may provide solutions to some of the following more general open problems.

**Problem 3.** Is there a non-finitely based variety generated by a \(*\)-semigroup of order five or less?

**Problem 4.** Is there a variety with \(2^{\aleph_0}\) subvarieties generated by a \(*\)-semigroup of order five or less?

Problems 3 and 4 are motivated by the existence of non-finitely based varieties generated by a \(*\)-semigroup of order six [3, 10, 23], some of which have \(2^{\aleph_0}\) subvarieties [24].

The variety of all commutative semigroups is long known to be finitely universal [4]. It follows that the variety of all commutative \(*\)-semigroups is also finitely universal; however, this variety is not generated by a finite \(*\)-semigroup.

**Problem 5.** Is there a finitely universal variety generated by a finite \(*\)-semigroup?

As for finitely universal varieties of semigroups, besides \(VA_2\) and \(VB_2\), there exist several that are generated by a semigroup with as few as four elements [17].

Since the \(*\)-semigroup \(\langle A_2, * \rangle\) is currently the smallest one known to generate a non-Specht variety, the existence of a smaller example is also of interest.

**Problem 6.** Is there a non-Specht variety generated by a \(*\)-semigroup of order four or less?

Every semigroup of order four or less generates a Specht variety, but it is unknown if the same result holds for all semigroups of order five [21].

2. **Preliminaries**

Acquaintance with rudiments of universal algebra is assumed. Refer to the monograph of Burris and Sankappanavar [5] for more information.
2.1. Terms and words. Let \( \mathcal{X} \) be a countably infinite alphabet throughout and let \( \mathcal{X}^* = \{ x^* \mid x \in \mathcal{X} \} \) be a disjoint copy of \( \mathcal{X} \). Elements of \( \mathcal{X} \cup \mathcal{X}^* \) are called variables. The free \( * \)-monoid over \( \mathcal{X} \) is the free semigroup \((\mathcal{X} \cup \mathcal{X}^*)^*\), together with the empty string \( \varepsilon \), using unary operation \( * \) given by \( (x^n)^* = x^n \) for all \( x \in \mathcal{X} \),

\[
(x_1 x_2 \cdots x_m)^* = x_m^* x_{m-1}^* \cdots x_1^*
\]

for all \( x_1, x_2, \ldots, x_m \in \mathcal{X} \cup \mathcal{X}^* \cup \{ \varepsilon \} \). \( \varepsilon^* = \varepsilon \). Elements of the \( * \)-monoid \((\mathcal{X} \cup \mathcal{X}^*)^* \cup \{ \varepsilon \} \) are called terms, while words in the monoid \((\mathcal{X} \cup \mathcal{X}^*)^* \cup \{ \varepsilon \} \) are said to be plain. The plain projection of a word \( w \in (\mathcal{X} \cup \mathcal{X}^*)^* \), denoted by \( \overline{w} \), is the plain word obtained from \( w \) by removing all occurrences of the symbol \( * \). The content of a word \( w \), denoted by \( \text{con}(w) \), is the set of variables occurring in \( w \). If \( x, x' \in \text{con}(w) \) for some \( x \in \mathcal{X} \), then \( \{ x, x' \} \) is called a mixed pair of \( w \).

The set of terms over \( \mathcal{X} \) is the smallest set \( T(\mathcal{X}) \) such that
\[
\begin{align*}
\bullet & \quad \mathcal{X} \cup \{ \varepsilon \} \subseteq T(\mathcal{X}); \\
\bullet & \quad \text{if } t_1, t_2 \in T(\mathcal{X}), \text{ then } t_1 t_2 \in T(\mathcal{X}); \\
\bullet & \quad \text{if } t \in T(\mathcal{X}), \text{ then } t^* \in T(\mathcal{X}).
\end{align*}
\]

The subterms of a term \( t \) are then recursively defined as follows:
\[
\begin{align*}
\bullet & \quad \text{if } t \text{ is a subterm of } t; \\
\bullet & \quad \text{if } s_1, s_2 \text{ are subterms of } t \text{ where } s_1, s_2 \in T(\mathcal{X}), \text{ then so are } s_1 \text{ and } s_2; \\
\bullet & \quad \text{if } s^* \text{ is a subterm of } t \text{ where } s \in T(\mathcal{X}), \text{ then so is } s.
\end{align*}
\]

The proper inclusion \((\mathcal{X} \cup \mathcal{X}^*)^* \subseteq T(\mathcal{X}) \) holds and the involution axioms can be used to convert any term \( t \in T(\mathcal{X}) \) to \( \{ \varepsilon \} \) into a unique word \([t] \in (\mathcal{X} \cup \mathcal{X}^*)^* \). For instance, \([x (x^3 (yx^n)^*)^* y^r y^s] = xy (x^r y^n)^* y^r y^s\].

Remark 7. For any subterm \( s \) of a term \( t \), either \([s] \) or \([s]^* \) is a factor of \([t] \).

2.2. Equations, deducibility, and satisfiability. An equation is an expression \( s \approx t \) formed by terms \( s, t \in T(\mathcal{X}) \setminus \{ \varepsilon \} \). More specifically, a word equation is an equation \( u \approx v \in (\mathcal{X} \cup \mathcal{X}^*)^* \) and a plain equation is an equation \( u \approx v \) formed by plain words \( u, v \). An equation \( s \approx t \) is directly deducible from an equation \( p_1 \approx p_2 \) if there exists some substitution \( \alpha : \mathcal{X} \rightarrow T(\mathcal{X}) \setminus \{ \varepsilon \} \) such that \( \alpha(p_1) \) is a subterm of \( s \) and replacing this particular subterm \( \alpha(p_1) \) of \( s \) with \( \alpha(p_2) \) results in the term \( t \). An equation \( s \approx t \) is deducible from a set \( \Sigma \) of equations if there exists a finite sequence \( s = t_1, t_2, \ldots, t_r = t \) of distinct terms such that each equation \( t_i \approx t_{i+1} \) is directly deducible from some equation in \( \Sigma \).

A \( * \)-semigroup \( (S, * ) \) satisfies an equation \( s \approx t \), or \( s \approx t \) is satisfied by \( (S, * ) \), if for any substitution \( \alpha : \mathcal{X} \rightarrow S \), the elements \( \alpha(s) \) and \( \alpha(t) \) of \( S \) coincide; in this case, \( s \approx t \) is also said to be an equation of \( (S, * ) \). For any class \( \mathfrak{R} \) of \( * \)-semigroups, the set of equations satisfied by every \( * \)-semigroup in \( \mathfrak{R} \), denoted by \( \text{Eq} \mathfrak{R} \), is called the equational theory of \( \mathfrak{R} \). An equational basis for \( \mathfrak{R} \) is any subset \( \Sigma \) of \( \text{Eq} \mathfrak{R} \) such that every equation of \( \mathfrak{R} \) is deducible from \( \Sigma \). A variety is finitely based if it has a finite equational basis. A Specht variety is a variety whose subvarieties are all finitely based.

2.3. Certain equations of \( \langle A_2, * \rangle \). For each \( n \geq 2 \), define the sets of words
\[
\begin{align*}
P_n &= \{ x y_1^r y_2^s \cdots y_n^r x^* \mid r_1, r_2, \ldots, r_n \geq 2 \} \quad \text{and} \\
P_n^* &= \{ (y_n^r)^* (y_{n-1}^r)^* \cdots (y_1^r)^* x^* \mid r_1, r_2, \ldots, r_n \geq 2 \}.
\end{align*}
\]

Note that \( P_n^* = \{ [u]^* \mid u \in P_n \} \).

Lemma 8. Let \( u \approx v \) be any word equation such that \( u \in P_n \cup P_n^* \). Suppose that \( u \approx v \in \text{Eq}(A_2, * ) \). Then
(i) \( v \) begins with \( x \) and ends with \( x^* \);

(ii) \( \text{con}(\nabla) = \text{con}(\Pi) = \{x, y_1, y_2, \ldots, y_n\} \);

(iii) none of the following words is a factor of \( v \):

\begin{itemize}
  \item (1) \( x^2, (x^*)^2, xx^*, x^*x \);
  \item (2) \( y_iy_j, y_i'y_j', \text{where } 1 \leq i, j \leq n \text{ and } j \neq i + 1 \);
  \item (3) \( y_iy_j^*, \text{where } 1 \leq i, j \leq n \);
  \item (4) \( y_i'y_j, \text{where } 1 \leq i, j \leq n \);
  \item (5) \( x^*y_i', y_ix, \text{where } 1 \leq i \leq n \);
  \item (6) \( x^*y_i, y_i'x^*, \text{where } 1 \leq i \leq n \);
  \item (7) \( xy_i, y_i'x^*, \text{where } 2 \leq i \leq n \);
  \item (8) \( xy_i', y_ix^*, \text{where } 1 \leq i \leq n - 1 \).
\end{itemize}

Proof. The substitutions \( \alpha, \beta, \gamma_m : \mathcal{X} \to A_2 \), where \( z \in \mathcal{X} \) and \( m \in \{1, 2, \ldots, 8\} \), are required in this proof:

\[
\alpha(t) = \begin{cases}
  a & \text{if } t = x, \\
  e & \text{otherwise;}
\end{cases}
\]

\[
\beta_z(t) = \begin{cases}
  0 & \text{if } t = z, \\
  e & \text{otherwise;}
\end{cases}
\]

\[
\gamma_1(t) = \begin{cases}
  a & \text{if } t = x, \\
  e & \text{otherwise;}
\end{cases}
\]

\[
\gamma_2(t) = \begin{cases}
  ea & \text{if } t = y_1, \\
  e & \text{otherwise;}
\end{cases}
\]

\[
\gamma_3(t) = \begin{cases}
  e & \text{if } t = x, \\
  ea & \text{otherwise;}
\end{cases}
\]

\[
\gamma_4(t) = \begin{cases}
  e & \text{if } t = x, \\
  a & \text{otherwise;}
\end{cases}
\]

\[
\gamma_5(t) = \begin{cases}
  a & \text{if } t = x, \\
  e & \text{otherwise;}
\end{cases}
\]

\[
\gamma_6(t) = \begin{cases}
  a & \text{if } t = y_i, \\
  e & \text{otherwise;}
\end{cases}
\]

\[
\gamma_7(t) = \begin{cases}
  e & \text{if } t = x, \\
  a & \text{otherwise;}
\end{cases}
\]

\[
\gamma_8(t) = \begin{cases}
  e & \text{if } t = y_i, \\
  a & \text{otherwise.}
\end{cases}
\]

(i) If either \( v \) does not begin with \( x \) or \( v \) does not end with \( x^* \), then the contradiction \( \alpha(u) = a \neq \alpha(v) \) is obtained.

(ii) If there exists some variable \( z \) of \( \Pi \) that does not occur in \( v \), then the contradiction \( \beta_z(u) = 0 \neq e = \beta_z(v) \) is obtained. Therefore the inclusion \( \text{con}(\Pi) \subseteq \text{con}(v) \) holds. By symmetry, the reverse inclusion \( \text{con}(\Pi) \supseteq \text{con}(v) \) also holds.

(iii) If for some \( m \in \{1, 2, \ldots, 8\} \), a word from \( (m) \) is a factor of \( v \), then the contradiction \( \gamma_m(u) \neq 0 = \gamma_m(v) \) is obtained.

\[\square\]

Lemma 9. Let \( u \approx v \) be any equation such that \( u \in \mathcal{P}_n \cup \mathcal{P}_n^* \) and \( v \in (\mathcal{X} \cup \mathcal{X}^*)^+ \). Then \( u \approx v \in \text{Eq}(A_2, *) \) if and only if \( v \in \mathcal{P}_n \cup \mathcal{P}_n^* \).

Proof. Suppose that \( u \approx v \in \text{Eq}(A_2, *) \). Then by Lemma 8(i, ii), one has \( v = xwz^* \) for some \( w \in (\mathcal{X} \cup \mathcal{X}^*)^+ \) with \( \text{con}(w) = \{x, y_1, y_2, \ldots, y_n\} \). By Lemma 8(iii), the factors of \( v \) of length two can only possibly be

\[
x y_1 y_1^* y_2^* y_2 y_2^* \ldots y_n^* y_n y_n^* y \in A_n^*, \ y_1^* y_2^* \ldots y_n^* x^*.
\]

It follows that the word \( v \) is either

\[
x y_1 y_2^* \ldots y_n^* x^* \text{ or } x(y_n^*)r_n \ldots (y_1^*)r_1 x^*
\]

for some \( r_1, r_2, \ldots, r_n \geq 1 \). Define the substitution \( \alpha_i : \mathcal{X} \to A_2 \) by

\[
\alpha_i(t) = \begin{cases}
  a & \text{if } t = y_i, \\
  e & \text{otherwise.}
\end{cases}
\]
Then whenever \( r_i = 1 \), the contradiction \( \alpha_i(u) = 0 \neq e = \alpha_i(v) \) is obtained. Therefore \( r_1, r_2, \ldots, r_n \geq 2 \), whence \( v \in \mathcal{P}_n \cup \mathcal{P}_n^* \).

Conversely, if \( v \in \mathcal{P}_n \cup \mathcal{P}_n^* \), then it is easily shown that the equation \( u \approx v \) is deducible from the equations \( \{ x^2 \approx x^2, xyx^* \approx (xyx^*)^*, (x^2)^* x = (xy)^* y^* x^* \} \) of \( (A_2, *) \).

\[ \square \]

3. AN INFINITE DESCENDING CHAIN

For each \( n \geq 2 \), let \( W_n \) denote the subvariety of \( V(A_2, *) \) defined by the equation \( p_n \approx q_n \), where

\[ p_n = xy_1y_2^2 \cdots y_{n-1}^2y_n^2x^* \quad \text{and} \quad q_n = xy_ny_{n-1}^2 \cdots y_1^2x^*, \]

and let \( W = \bigcap_{n \geq 2} W_n \). Note that \( p_n \in \mathcal{P}_n \) but \( q_n \notin \mathcal{P}_n \cup \mathcal{P}_n^* \). The \(*\)-semigroup \( (B_2, *) \) has commuting idempotents and so satisfies the equation \( p_n \approx q_n \).

Therefore \( W_n \) and \( W \) are varieties in the interval \( [V(B_2, *), V(A_2, *)] \). Further, the inclusion \( W_{n+1} \subseteq W_n \) holds because the equation \( p_n \approx q_n \) of \( W_n \) is deducible from the equations \( \{ x^3 \approx x^2, p_{n+1} \approx q_{n+1} \} \) of \( W_{n+1} \); specifically,

\[ p_n \approx xy_1y_2^2 \cdots y_{n-1}^2y_n^2x^* \quad \text{by} \ x^3 \approx x^2 \]
\[ \approx xy_ny_{n-1}^2 \cdots y_1^2x^* \quad \text{by} \ p_{n+1} \approx q_{n+1} \]
\[ \approx q_n \quad \text{by} \ x^3 \approx x^2. \]

Consequently, the inclusions

\[ V(A_2, *) \supseteq W_2 \supseteq W_3 \supseteq \cdots \supseteq W \supseteq V(B_2, *) \]

hold. These inclusions are shown to be proper in Inequalities 1–4 below.

**Inequality 1.** \( V(A_2, *) \neq W_2. \)

**Proof.** Since \( p_2 \in \mathcal{P}_2 \) and \( q_2 \notin \mathcal{P}_2 \cup \mathcal{P}_2^* \), it follows from Lemma 9 that \( \langle A_2, * \rangle \) does not satisfy \( p_2 \approx q_2 \). \( \square \)

**Lemma 10.** For any substitution \( \alpha : \mathcal{X} \to T(\mathcal{X}) \setminus \{ \emptyset \} \), none of the words \( [\alpha(p_n)] \), \( [\alpha(q_n)] \), and \( [\alpha(q_n)] \) can be a factor of any word in \( \mathcal{P}_{n+1} \cup \mathcal{P}_{n+1}^* \).

**Proof.** Observe that

- \( p_n \) and \( q_n \) can be obtained from one other by renaming of variables;
- if \( u, v_1, v_2, \ldots, v_n \in T(\mathcal{X}) \setminus \{ \emptyset \} \) are such that

\[ \alpha(x) = u, \ \alpha(y_1) = v_1, \ \alpha(y_2) = v_2, \ldots, \ \alpha(y_n) = v_n, \]

then \( [(\alpha(p_n))^*] = [\beta(p_n)] \) and \( [(\alpha(q_n))^*] = [\beta(q_n)] \), where \( \beta \) is any substitution such that

\[ \beta(x) = u, \ \beta(y_1) = v_n^*, \ \beta(y_2) = v_{n-1}^*, \ldots, \ \beta(y_n) = v_1^*. \]

- the words in \( \mathcal{P}_{n+1} \) and the words in \( \mathcal{P}_{n+1}^* \) can be obtained from one other by renaming of variables.

Therefore by symmetry, it suffices to show that \( [\alpha(p_n)] \) is not a factor of any word in \( \mathcal{P}_{n+1} \). Seeking a contradiction, suppose that \( [\alpha(p_n)] \) is a factor of some word in \( \mathcal{P}_{n+1} \), say \( u[\alpha(p_n)]v \in \mathcal{P}_{n+1} \) for some \( u, v \in (\mathcal{X} \cup \mathcal{X}^*)^+ \cup \{ \emptyset \} \). Then

\[ u[\alpha(x)] [\alpha(y_1)]^2 [\alpha(y_2)]^2 \cdots [\alpha(y_n)]^2 [\alpha(x)]^2 v = xy_1y_2^2 \cdots y_{n+1}^2x^* \]

for some \( r_1, r_2, \ldots, r_{n+1} \geq 2 \). Since \( \{ x, x^* \} \) is the only mixed pair of the word on the right side, simple inspection shows that

- \( u = \emptyset \) and \( [\alpha(x)] \) is a nonempty prefix of \( xy_1y_2^2 \cdots y_{n+1}^2 \);
- \( v = \emptyset \) and \( [\alpha(x)] \) is a nonempty suffix of \( y_1y_2^2 \cdots y_{n+1}^2 x^* \);
- \( [\alpha(y_1)]^2 [\alpha(y_2)]^2 \cdots [\alpha(y_n)]^2 \) is a nonempty factor of \( y_1y_2^2 \cdots y_{n+1}^2 \).
If \(|\alpha(x)| \neq x\), then \([\alpha(x)] = xy_1 \cdots\) by (a) and so \([\alpha(x)^*] = \cdots y_1^* x^*\), which contradicts (b). Therefore \([\alpha(x)] = x\) and \([\alpha(x)^*] = x^*\), whence by (c),
\[
[\alpha(y_1)]^2[\alpha(y_2)]^2 \cdots [\alpha(y_n)]^2 = y_1^2 y_2^2 \cdots y_n^{r_n+1}.
\]

By comparing the number of distinct variables on both sides of this equality, it is obvious that some word \([\alpha(y_i)]\) on the left is a plain word with at least two distinct variables and so contains some \(y_j y_{j+1}\) as a factor. However, the left side would then be \(\cdots [\alpha(y_i)]^2 \cdots = \cdots y_j y_{j+1} \cdots\), and this is impossible in view of the right side.

\[\square\]

Inequality 2. \(W_n \neq W_{n+1}\).

**Proof.** Seeking a contradiction, suppose that \(W_n = W_{n+1}\), so that the equation \(p_{n+1} \approx q_{n+1}\) of \(W_{n+1}\) is deducible from \(\text{Eq}W_n\). Then there exists a finite sequence
\[p_{n+1} = t_1, t_2, \ldots, t_r = q_{n+1}\]
of distinct terms, where each equation \(t_i \approx t_{i+1}\) is directly deducible from some equation in \(\text{Eq}(A_2, *) \cup \{p_i \approx q_i\}\). It is clear that \(\{t_1\} = p_{n+1} \in \mathcal{P}_{n+1} \cup \mathcal{P}_{n+1}^*\).

Suppose that \(\{t_i\} \in \mathcal{P}_{n+1} \cup \mathcal{P}_{n+1}^*\) for some \(i \geq 1\). If there exists a substitution \(\alpha : \mathcal{X} \to \mathcal{T}(\mathcal{X}) \setminus \{\mathcal{B}\}\) such that either \(\alpha(p_i)\) or \(\alpha(q_i)\) is a subterm of \(t_i\), then as observed in Remark 7, one of \([\alpha(p_i)], [\alpha(p_i)]^r, [\alpha(q_i)],\) and \([\alpha(q_i)]^r,\) is a factor of \(\{t_i\} \in \mathcal{P}_{n+1} \cup \mathcal{P}_{n+1}^*\), whence Lemma 10 is contradicted. Therefore the equation \(t_i \approx t_{i+1}\) is not directly deducible from \(p_i \approx q_i\) and so is directly deducible from some equation in \(\text{Eq}(A_2, *)\). It follows that \(\{t_i\} \approx \{t_{i+1}\}\) is a word equation of \(\langle A_2, \ast \rangle\), whence \(\{t_{i+1}\} \in \mathcal{P}_{n+1} \cup \mathcal{P}_{n+1}^*\) by Lemma 9. Thus by induction, \(\{t_1\}, \{t_2\}, \ldots, \{t_r\} \in \mathcal{P}_{n+1} \cup \mathcal{P}_{n+1}^*\). But then \(q_{n+1} = \{t_r\} \in \mathcal{P}_{n+1} \cup \mathcal{P}_{n+1}^*\) is a contradiction.

\[\square\]

Inequality 3. \(W_n \neq W\).

**Proof.** As shown in the proof of Inequality 2, the equation \(p_{n+1} \approx q_{n+1}\) of \(W\) is not deducible from the equations of \(W_n\).

\[\square\]

Inequality 4. \(W \neq \mathcal{V}(B_2, \ast)\).

**Proof.** Seeking a contradiction, suppose that \(W = \mathcal{V}(B_2, \ast)\), so that the equation \(x^2 y^2 \approx y^2 x^2\) of \(\mathcal{V}(B_2, \ast)\) is deducible from \(\text{Eq}W\). More specifically, the equation \(x^2 y^2 \approx y^2 x^2\) is deducible from some finite subset \(\Sigma\) of \(\text{Eq}(A_2, \ast) \cup \{p_i \approx q_i \mid i \geq 2\}\).

Since \(\langle A_2, \ast \rangle\) does not satisfy \(x^2 y^2 = y^2 x^2\), the set \(\Sigma\) must contain some \(p_i \approx q_i\). Let \(n \geq 2\) be the largest integer such that \(p_n \approx q_n \in \Sigma\). Then \(W_n\) satisfies \(\Sigma\), it satisfies \(x^2 y^2 = y^2 x^2\) and so also \(p_{n+1} \approx q_{n+1}\). It follows that \(W_n = W_{n+1}\), whence Inequality 2 is contradicted.

\[\square\]

**Corollary 11.** The variety \(W\) is non-finitely based.

**Proof.** Seeking a contradiction, suppose that \(W\) is finitely based, say some finite subset \(\Sigma\) of \(\text{Eq}(A_2, \ast) \cup \{p_i \approx q_i \mid i \geq 2\}\) is an equational basis. If \(\Sigma\) does not contain any \(p_i \approx q_i\), then \(\mathcal{V}(A_2, \ast)\) satisfies \(\Sigma\) and so coincides with \(W\), which is impossible. Therefore \(\Sigma\) must contain some \(p_i \approx q_i\). Let \(n \geq 2\) be the largest integer such that \(p_n \approx q_n \in \Sigma\). Then \(W_n\) satisfies \(\Sigma\), so that \(W_n = W\), whence Inequality 2 is contradicted.

\[\square\]

**Acknowledgments**

The author thanks the anonymous reviewer for a number of helpful suggestions.
References


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